THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fifty-fifth regular meeting of the Society was held in New York City on Saturday, October 28, extending through the usual morning and afternoon sessions. The attendance included the following thirty-five members:

Professor W. J. Berry, Professor G. D. Birkhoff, Professor G. A. Bliss, Professor E. W. Brown, Professor B. H. Camp, Professor F. N. Cole, Dr. G. M. Conwell, Dr. H. B. Curtis, Dr. L. L. Dines, Professor L. P. Eisenhart, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Professor W. A. Garrison, Professor C. C. Grove, Professor E. V. Huntington, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. N. J. Lennes, Professor James Maclay, Dr. Emilie N. Martin, Dr. H. H. Mitchell, Dr. Anna J. Pell, Dr. H. W. Reddick, Professor R. G. D. Richardson, Miss S. F. Richardson, Professor Paul Saurel, Mr. L. P. Siceloff, Professor P. F. Smith, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. H. E. Webb, Miss E. C. Williams.

President Fine occupied the chair at the two sessions. The Council announced the election of the following persons to membership in the Society: Professor T. B. Ashcraft, Colby College; Professor Clara L. Bacon, Goucher College; Professor J. M. Davis, State University of Kentucky; Professor W. C. Eells, Whitworth College; Dr. J. L. Jones, Yale University; Professor F. C. Kent, University of Oklahoma; Professor L. C. Plant, University of Montana; Mr. R. E. Powers, Denver, Colo.; Mr. T. M. Simpson, University of Wisconsin; Professor Evan Thomas, University of Vermont; Professor H. C. Wolff, University of Wisconsin; Mr. W. A. Zehring, Purdue University. Nine applications for membership in the Society were received.

A list of nominations for officers and other members of the Council, to be placed on the official ballot for the annual meeting, was adopted. Committees were appointed to make arrangements for the summer meeting of the Society at the University of Pennsylvania, in 1912, and to audit the Treasurer's accounts. The invitation of the University of Wisconsin to hold the summer meeting and colloquium at that University in 1913 was accepted.

A committee consisting of President Fine as chairman and Professors E. W. Brown, C. J. Keyser, E. H. Moore, W. F. Osgood, and E. B. Van Vleck was appointed to consider and report to the Council a plan for placing the business of the Society on a permanent basis. It was decided to change the form of the Annual Register by omitting in the personal entries all mention of membership in other organizations.

The following papers were read at this meeting:

(1) Mr. A. R. SCHWEITZER: "On a functional equation."

(2) Professor E. V. HUNTINGTON: "A new approach to the theory of relativity."

(3) Mr. L. P. SICELOFF: "Simple groups from order 2,001 to order 3,640."

(4) Dr. H. H. MITCHELL: "Determination of the quaternary linear groups by geometrical methods."

(5) Professor G. A. BLISS: "A new proof of the existence theorem for implicit functions."

(6) Mr. R. E. POWERS: "The tenth perfect number."

(7) Professor E. W. BROWN: "On the summation of a certain triply infinite series."

(8) Dr. L. L. DINES: "On the highest common factor of a system of polynomials."

(9) Professor R. D. CARMICHAEL: "Fundamental properties of a reduced residue system mod m."

(10) Professor R. D. CARMICHAEL: "A generalization of Cauchy's functional equation."

(11) Professor R. D. CARMICHAEL: "On composite numbers P which satisfy the Fermat congruence $a^{P-1} \equiv 1 \bmod P$."

(12) Professor EDWARD KASNER: "Differential invariants of infinite order."

(13) Professor B. H. CAMP: "Series of Laplace's functions."

(14) Dr. N. J. LENNES: "A new proof that a Jordan curve separates a plane."

In the absence of the authors the papers of Mr. Schweitzer, Mr. Powers, and Professor Carmichael were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Schweitzer proves the following theorem:

Let $u = \phi(x, z)$, $v = \phi(x, y)$ and $w = \phi(y, z)$; then if $\phi(u, v) = w$, there exists a function χ such that $\chi(v) = \chi(x) - \chi(y)$. A similar result is valid in case of the relation $\phi(u, w) = v$.

The method of proof is analogous to that of Abel * and Stäckel † who show (in effect) that if $\phi(y, u) = \phi(z, v) = \phi(x, w)$, then there exists a function χ such that $\chi(v) = \chi(x) + \chi(y)$.

2. The theory of relativity as developed by Einstein ‡ is usually supposed to involve a radical modification, not only of our conception of the ether, but also of our fundamental notions concerning space and time; and the discussion of the so-called "paradoxes of relativity" has often led beyond the safe ground of mathematical deduction into the realm of metaphysical speculation.

The purpose of Professor Huntington's article is to show that the famous "transformation equations," which stand at the center of the theory, can be easily deduced from simple conventions in regard to the setting of clocks and the laying out of coordinate systems, without any conflict with our ordinary conception of the ether, or with our ordinary notions of space and time, thereby freeing the theory from the least appearance of paradox.

The method pursued is in some sense a return to the point of view of Lorentz,§ who retained the concept of the ether as his starting point, and never abandoned our ordinary notions of time and space; and the transformation equations here obtained resemble the equations of Lorentz in being slightly more general than those of Einstein. Lorentz's method of deducing these equations, however, involves a large and difficult part of the modern electromagnetic theory, while the method here adopted depends only on the most elementary considerations.

The present article is purely expository, and does not attempt any critical or historical discussion.‖

3. In this paper Mr. Siceloff continues the search for new simple groups of low order. He finds that the only orders of simple groups between 2,001 and 3,640 are those already well-

* *Crelle's Journal*, vol. 1 (1826), p. 11.

† *Zeitschrift für Mathematik und Physik*, vol. 42 (1897), p. 323.

‡ A. Einstein, "Zur Elektrodynamik bewegter Körper," *Annalen der Physik*, ser. 4, vol. 17 (1905), pp. 891–921.

§ H. A. Lorentz, Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern, Leiden, 1895. Neudruck, Leipzig, 1906. Also, The Theory of Electrons (Columbia University lectures, 1906), Leipzig, 1909.

‖ For an extensive bibliography of the theory of relativity, the reader is referred to an article by J. Laub, in the *Jahrbuch der Radioaktivität und Elektronik*, vol. 7 (1910), December, p. 405.

known, viz., 2,520 and 3,420. Whether there is a second simple group of order 2,520 has not been determined. The results are based chiefly on the study of the Sylow subgroups.

4. A complete enumeration of all primitive collineation groups in ordinary space of three dimensions has been given by Blichfeldt (*Mathematische Annalen*, 1905). This was done primarily by algebraic methods and involved the use of theorems concerning the irreducibility of equations containing roots of unity. It is the purpose of Dr. Mitchell's paper to determine these groups by geometrical methods and in such a way as to determine also the quaternary groups where the coefficients of the transformations are marks of the $GF(p^n)$, but the order of the group is not divisible by the prime modulus p. Properties of the linear complex are found to be of much service in the solution of the problem.

5. The paper of Professor Bliss has to do with the solution of a system of equations of the form

$$f_i(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n) = 0 \quad (i = 1, 2, \cdots, n).$$

The theorem stating the existence of a set of solutions of the form

$$y_i = y_i(x_1, x_2, \cdots, x_m) \quad (i = 1, 2, \cdots, n)$$

for these equations is an old one, but in the well-known proofs of it a single equation is usually considered first, and then the proof is made for a system of equations by applying the principle of mathematical induction. In the present paper a proof is given which seems relatively simple, and which applies with equal convenience to either one or more equations.

6. The composition of Mersenne's numbers $2^p - 1$, p a prime, is known for all values of p not exceeding 100 with the exception of $p = 89$. E. Lucas, in volume 1 (1878) of the *American Journal of Mathematics*, proved the following theorem: "If $N = 2^{4q+1} - 1$, and if we form the series of residues (modulo N) 4, 14, 194, $\cdots$, each of which is equal to the square of the preceding, diminished by two units, the number N is prime if the first residue 0 lies between the $2q$th and the $(4q + 1)$th term." Mr. Powers shows that for $4q + 1 = 89$ the 88th term of the foregoing series is 0 (first 0); hence $2^{89} - 1$ is a prime number, and $2^{88}(2^{89} - 1)$ is the tenth perfect number.

7. In the expansion of Hill's infinite determinant there occurs a term which is equivalent to

$$\Sigma_i\Sigma_k\Sigma_{k'}\{i\}\{i+1\}\{i+k\}\{i+k+1\}\{i+k+k'\}\{i+k+k'+1\},$$

where each of the six factors is obtained by giving j the proper value in $1/\{j\} = \alpha^2 - j^2$. Here α is unequal to any integer, i takes all integral values from $+\infty$ to $-\infty$, and k, k' all integral values from 2 to ∞. Hill gives the value with but slight indication of the method by which he obtained it. Professor Brown shows how it may be obtained without great labor, and is thus able to verify Hill's result.

8. In a paper before the Chicago meeting of the Society in April, Dr. Dines considered the conditions characterizing the existence of a common factor of first or higher degree of a system of n polynomials in one variable. It was found that, if a certain restriction be made upon the coefficients, then necessary and sufficient conditions can be stated in terms of the vanishing of $n - 1$ determinants.

In the present paper, a matrix is constructed whose elements are the coefficients of the given polynomials arranged according to a simple rule, and which for $n = 2$ reduces to the matrix of the well known Sylvester resultant of two polynomials. This matrix possesses the following properties:

(1) The vanishing of the matrix constitutes necessary and sufficient conditions for the existence of a common factor of first or higher degree. (A matrix of μ rows and ν columns, $\mu \leqq \nu$, is said to vanish when every determinant of order μ of the matrix vanishes.)

(2) The degree of the highest common factor of the system of polynomials can be stated in terms of the rank of the matrix.

(3) The coefficients of the highest common factor are determinants of the matrix, easily characterized.

(4) There are always $n - 1$ determinants of the matrix whose vanishing is equivalent to the vanishing of the matrix.

9. The most important example of an abelian group is furnished by the $\varphi(m)$ integers of a reduced residue system mod m, the operation of combination being multiplication and reduction mod m. In the present paper Professor Carmichael determines the invariants of such a group and gives a method for constructing all groups of this class having the same in-

variants as a given group of the class. In this way all groups of the class, simply isomorphic with a given one, are determined. Let us define $\lambda(m)$ as the greatest exponent to which any element of the reduced residue system mod m belongs. A method is given for finding every x such that $\lambda(x) = \lambda(m)$, and incidentally for finding every x such that $\varphi(x) = \varphi(m)$. The paper closes with a table giving all possible solutions of $\lambda(x) = a$ for every possible value of a up to $a = 24$. The solutions are separated into sets according to the type of group defined by the corresponding reduced residue system. In each case the invariants of the group are given.

10. The nature of the generalization of Cauchy's functional equations effected in Professor Carmichael's second paper is indicated by the following theorem: If α is any positive or negative quantity and u_s is any function of z capable of assuming at least once each value in the set

$$2^{-s}n\alpha \quad (n, s = 0, 1, 2, \cdots;\ n \leqq 2^s),$$

then the most general function $f(x)$ which is continuous on the interval $(0, \alpha)$ and satisfies the functional equation

$$f(x + u_s) = f(x) + f(u_s)$$

is $f(x) = ax$, where a is an arbitrary constant.

11. In his note on the Fermat congruence $a^{P-1} \equiv 1 \bmod P$ Professor Carmichael gives a method of finding composite numbers P which satisfy the congruence for every a which is prime to P.

12. When an analytic curve is referred to the tangent and normal at one of its points, its equation takes the form

$$y = a_2x^2 + a_3x^3 + \cdots.$$

The coefficients are differential invariants of the curve, the first being half the curvature. Every combination of the coefficients is an invariant, that is, intrinsically related to the curve. Professor Kasner considers combinations involving an infinitude of coefficients. An important example is the radius of convergence of series y. Another invariant is the quantity J defined as the first non-vanishing term in the sequence $a_2, a_3, \cdots$. This

vanishes only when the curve is a straight line, and is applied to certain questions concerning geodesics and trajectories. The even and odd components of a curve at a given point, introduced by the author in an earlier paper, are examples of covariants of infinite order.

13. The most important theorem concerning the validity of the expansion of an arbitrary function in a series of Laplace's functions has been proved by Jordan in his Cours d'Analyse, second edition, volume 2, page 252. The conditions stated there are that the given function be continuous on the surface of the sphere within some small circle about the point at which the expansion is made, and that it have limited variation along every great circle through the point.

The object of Professor Camp's paper is to correct an error in Jordan's theorem, and to furnish new sufficient conditions for the validity of these expansions. To the conditions announced by Jordan should be added the requirements that the values of the variations be all less than some fixed number, and that these variations be uniform with respect to all great circles through the point. Furthermore, the conditions of continuity of the function and of uniformity of the variations may be replaced by the different condition that there exist a small circle about the point considered such that within it the given function is, along all great circles through the point, an indefinite integral of another function which has an absolutely convergent double Lebesgue integral in its domain of definition. Any or all of these conditions may fail on a null set of great circles through the point.

14. In a paper presented to the Society at the April meeting Dr. Lennes proved that a closed continuous surface separates a three-space in which it lies into two connected sets. The present paper is an adaptation of the methods used in that paper to the case of the curve in the plane.

F. N. COLE,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION.

THE twentieth regular meeting of the San Francisco Section of the American Mathematical Society was held at the University of California on Saturday, October 28, 1911. The following seventeen members of the society were present:

Professor R. E. Allardice, Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Mr. C. G. P. Kuschke, Professor A. O. Leuschner, Professor J. H. McDonald, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Professor E. W. Ponzer, Professor T. M. Putnam, Professor A. W. Whitney.

In the absence of Professor Lehmer, chairman of the section, Professor Haskell presided over the morning session and Professor Allardice over the afternoon session.

The following officers were elected for the ensuing year: president, Professor L. M. Hoskins; secretary, Professor T. M. Putnam; program committee, Professors Blichfeldt, McDonald, and Putnam.

The dates for the meetings for 1912 were fixed at April 6 and October 26.

The following papers were presented:

(1) Professor W. A. MANNING: "The primitive groups of class twelve."

(2) Professor G. A. MILLER: "Note on the maximal cyclic subgroups of a group of order p^m."

(3) Dr. H. W. STAGER: "A Sylow factor table of the first twelve thousand numbers."

(4) Professor A. W. WHITNEY: "Theory of schedule rating in fire insurance."

(5) Professor H. F. BLICHFELDT: "On a basis of plane geometry" (preliminary communication).

(6) Professor A. O. LEUSCHNER and Mr. B. A. BERNSTEIN: "Graphical solutions of the fundamental equations in the short methods of determining orbits."

In the absence of the authors the papers of Professor Miller and Dr. Stager were read by title. Abstracts of the papers follow below.

1. In addition to the 25 known primitive groups of class 12 on not more than 17 letters, Professor Manning has found that there are four others, one of degree 27, two of degree 28, and one of degree 36. Of these, only one is doubly transitive. It, and its maximal subgroup, which is primitive of degree 27 and of order $27 \cdot 1920$, seem not to have been noticed before.

2. It is known that every subgroup of order p^α in a group G of order p^m, $m > \alpha$ and p a prime number, is transformed into itself by at least $p^{\alpha+1}$ operators of G. Professor Miller calls a cyclic subgroup which is not contained in a larger cyclic subgroup of G a maximal cyclic subgroup of G, and proves the following related theorems:

A necessary and sufficient condition that every maximal cyclic subgroup of order p^α in a group G of order p^m, $m>3$, is transformed into itself by no more than $p^{\alpha+1}$ operators of G is that G contains one and only one cyclic subgroup of order p^{m-1}. Every non-cyclic group of order p^m, with the exception of the three of order 2^m which involve one and only one cyclic subgroup of order 2^{m-1}, contains at least one maximal cyclic subgroup of order p^α which is transformed into itself by more than $p^{\alpha+1}$ operators of the group.

3. The main purpose of the table described in detail in Dr. Stager's paper is to furnish directly information as to the possible number of subgroups of a group of given order under Sylow's well known theorem. Each number is expressed as the product of powers of primes, and for each prime factor greater than 2 there is given the values of k, other than zero, of all divisors of the number of the form $p(kp+1)$. Those values of k, other than zero, such that the number is identically equal to $p(kp+1)$ are specifically indicated. The table is so arranged that all of the above information appears for any number as one entry. In the case of primes, their rank is indicated by the symbol p_n. The desired possible number of subgroups is obtained by substituting the listed values of p and k in the formula $kp+1$.

In addition, a list of those numbers which contain no factors of the form $p(kp+1)$, $k>0$, is given, so arranged that the number of such numbers between any two limits less than 12,000 is easily obtained. The table was constructed independently by two different methods and the results compared for errors. The one method used is new and was probably applied for the first time in the construction of this table.

4. Since in fire insurance it is impossible to restrict the selection of risks to those that are standard, as is for the most part the case in life insurance, it becomes necessary to make an analysis of the hazard. A rate built up in this way is called a schedule rate. Mathematically the rate is a function of the various elements of construction, exposure, occupancy, etc., and the schedule is the expression of this relation. Existing schedules, while formed upon no real mathematical analysis, are as a matter of fact in the form of a Maclaurin expansion; a base rate is assumed and the departures from this are produced by additions. While this method works fairly well within a restricted field, it breaks down entirely, as might be expected, when the schedule is extended to a great variety of risks, and the consequence is that a great number of different schedules must be employed.

In the present paper Professor Whitney attempts by means of simplifying assumptions to make an analysis of the structure of this function. The result is a schedule no more complicated but radically different in structure. It is believed that a schedule of this form possesses a flexibility that makes it capable of universal application. The paper was originally given before the Actuarial Society of America and was published in the *Transactions* of that Society for May, 1911; rewritten in unmathematical form, it was published in the *Weekly Underwriter* of September 30.

5. In a paper published in the *Transactions* for 1902, pages 467–481, Professor Blichfeldt gave certain postulates which were sufficient to determine, for space of n dimensions, the typical forms for the distance between two points. In the present paper it is shown how for $n = 2$ a simpler set of postulates can be substituted. The existence of distance relations for four points is assumed only if one or two of the four points are one or both of two fixed points in the "plane." It is not assumed that the number of points of the plane is infinite.

6. In the preliminary determination of an orbit, the geocentric distance of the body can be obtained by solving the equation

$$(1) \qquad (z - p')^2 + q'^2 - \frac{h}{[(z-c)^2 + s^2]^{\frac{3}{2}}} = 0$$

if the orbit is assumed to be a parabola, and by solving

$$(z^2 - 2cz + 1)^3(z - m)^3 - m^2 = 0 \tag{2}$$

if no assumption is made regarding the orbit. One of the chief aims of Professor Leuschner's short methods for computing an orbit is to shorten the time taken for the calculation. The semi-graphical solution of the above equations used in the short methods contributes largely to this end, as well as to the lessening of tedious computational work; but this method still requires the use of tables. It is to eliminate entirely all computations in the solution of equation (1) above that Professor Leuschner and Mr. Bernstein devised methods of constructing geometrically the curve given by the equation

$$y = \frac{h}{[(z - c)^2 + s^2]^{\frac{1}{2}}},$$

whose intersections with the parabola $y = (z - p')^2$ give the required roots. By finding a geometric construction of the curve

$$z - m = \frac{m}{(z^2 - 2c + 1)^{\frac{1}{2}}},$$

whose intersections with the line $my = x + c - m$ give the roots of equation (2), Mr. Bernstein derived a purely graphical solution of the problem for the general orbit.

T. M. PUTNAM,
Secretary of the Section.

THE CARLSRUHE MEETING OF THE GERMAN MATHEMATICAL SOCIETY.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held in affiliation with the eighty-third convention of the society of naturalists and physicians at Karlsruhe during the week of September 24–28, under the presidency of Professor F. Schur, of the University of Strassburg. Under the stimulus of the pleasure and profit derived from preceding meetings a new record in attendance was established, over one hundred persons being present. The people of Karlsruhe generously

provided for the social part of the meeting by arranging an excellent program for every evening, one being devoted to an excursion to Heidelberg to witness the illumination of the castle, and to hear a well-rendered concert. The general session of Monday morning was opened with an address of welcome by the Grand Duke of Baden. Sectional meetings were held during the next four days, and the business meeting on Thursday morning. The membership of the society is 765 and the reserve fund consists of over M. 23,000. The bibliographic commission has been discontinued, and new members have been added to the committee to provide for the further publication of the papers of E. Schröder. Two finished volumes of the works of Euler were presented, and a detailed report of the German subcommittee of the international commission of mathematical instruction was read. Professors Rohn and Sommerfeld succeed Professors Engel and Schur as members of the council. The official representatives to the fifth congress of mathematicians are the new chairman, Professor W. von Dyck, and the secretary, Professor M. Krazer. The next meeting of the Vereinigung will be held at Münster in September, 1912, under the presidency of Professor von Dyck.*

The following papers were read at this meeting.

(1) Professor C. CARATHÉODORY, Breslau: "On the depiction of the boundary in the conformal representation of a general region."

(2) Professor O. BLUMENTHAL, Aachen: "Meromorphic functions of several variables."

(3) Professor G. FABER, Stuttgart: "On interpolation."

(4) Professor L. G. DU PASQUIER, Neuenburg: "On complex tettarions."

(5) Dr. L. LICHTENSTEIN, Berlin: "On Poisson's integral and the behavior of analytic functions on the boundary of the circle of convergence."

(6) Professor L. SCHLESINGER, Giessen: "On early work of Gauss on the arithmetic-geometric mean."

(7) Professor K. HEUN, Karlsruhe: "Outlines for the extension of classical mechanics" (report).

(8) Professor P. VORONETZ, Kieff: "On the equations of motion of a rigid body."

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*For this part of the report the editors are indebted to Dr. H. MOHRMANN, of Karlsruhe; for that of the social activities, to Dr. CLARA A. SMITH. The abstracts of the paper were kindly furnished by the authors in reply to a request sent by the editors.

(9) Professor R. v. MISES, Strassburg: "On the problems of oscillation and the Sommerfeld theorem of turbulence."

(10) Professor G. HAMEL, Brünn: "On the stiffness of ropes."

(11) Professor V. VARICAK, Agram: "On the non-euclidean interpretation of the theory of relativity."

(12) Professor E. PAPPERITZ, Freiberg: "Exhibition of the completed kinodiaphragmatic projection apparatus for geometric illustrations."

(13) Professor H. WIENER, Darmstadt: "Determination of a root of an equation by means of a convergent (or divergent) series."

(14) Professor E. SALKOWSKI, Charlottenburg: "Theory of curves in elliptic space."

(15) Professor C. JUEL, Copenhagen: "Concerning the straight lines which lie on an analytic non-algebraic surface of order three."

(16) Professors F. KLEIN, Göttingen, E. BROUWER, Amsterdam, and P. KOEBE, Leipzig: "Report on automorphic functions and the problem of uniformization."

(17) Professor E. HILB, Würzburg: "Connection between fundamental regions and linear differential equations."

(18) Dr. L. BIEBERBACH, Königsberg: "Remarks on automorphic functions."

(19) Professor J. WELLSTEIN, Strassburg: "On motion in space and the principle of relativity."

(20) Professor L. HEFFTER, Freiburg: "On the introduction of the four-dimensional world of Minkowski."

(21) Professor F. BERNSTEIN, Göttingen: "On Lagrange's problem of mean motion in the theory of secular perturbations."

(22) Professor H. WIENER, Darmstadt: "Exhibition and explanation of some new models."

Abstracts of several of the papers follow; the numbers correspond to those in the list above.

2. As an extension of memoirs by E. E. Levi, Professor Blumenthal discussed the natural boundaries of functions of two complex variables. He showed by means of the principle of condensation that the natural boundaries approach certain three-dimensional manifolds, many of the properties of which can be ascertained. The memoir is to appear in the *Jahresbericht*.

3. Professor Faber spoke of his investigations concerning

the accuracy of the best formulas of interpolation, and pointed out that they were less applicable to the representation of a function than to obtaining numerical approximations in mechanical quadrature. At the conclusion of the discussion the methods of Tchebichef and of Cauchy were compared.

4. The tettarions discussed by Professor Du Pasquier are defined as matrices of numbers having the same number of rows as of columns. They are added or subtracted by adding or subtracting corresponding components. Multiplication follows the same laws as the composition of linear substitutions. As subgroups, the complex numbers of Gauss and the quaternions of Hamilton are included. If $\nu = 4$ and all the components are real rational numbers, a tettarion field is defined. Division into integral and fractional forms can be made in an unlimited number of ways. Thus an infinite number of realms of rationality exist. A rational basis of classification is into integral, conditionally fractional, unconditionally fractional, and finally forms having both characters.

5. In the paper of Dr. Lichtenstein a number of properties of the partial derivatives of the Poisson integral, taken on the circumference, are derived. It is then shown that if $u(x, y)$ is a Poisson integral with integrable boundary function and $v(x, y)$ its complementary potential function, then the latter is integrable. The most important result concerns the partial derivatives of the second order of the logarithmic potential

$$u(x, y) = \frac{1}{4\pi}\int\int_T \log\{(x-\xi)^2 + (y-\eta)^2\}\, P(\xi, \eta)\, d\xi d\eta$$

in which $P(\xi, \eta)$ is an integrable function and T any finite region. It is shown that except for a set of points of content zero, the derivatives $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$ exist, are integrable, and satisfy the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = P(x, y).$$

8. The differential equations of motion of a rigid body are usually referred to a fixed system O_1-$x_1y_1z_1$ of rectangular coordinates, or to a system O-xyz moving rigidly with the body. In the few cases in which a system M-$\xi\eta\zeta$ is employed which is movable as to both, M is either supposed fixed, or is taken as the center of gravity of the moving body.

In Professor Voronetz's paper, the first part is concerned with the equations of motion in M-$\xi\eta\zeta$ coordinates, the position of M being unrestricted; the second part develops the case of rolling motion further, and contains a number of new results.

11. Professor Varicak showed in his paper that the Lobachevsky geometry furnishes adequate means for the treatment of the theory of relativity; all of the investigations operate in a Lobachevsky space. By means of simple considerations, the addition theory for velocities is developed, as well as the laws of aberration, of reflection in moving mirrors, and certain other laws.

12. Professor Papperitz presented an improved complete form of his apparatus for geometric illustrations. For the principle of the procedure and its applicability for producing figures in space for the purposes of instruction, reference may be made to two short reports ("Ueber das Zeichnen in Raume," *Jahresbericht*, volume 19 (1911), page 307, and "Die kinodiaphragmatische Projection . . . ," *Hoffmann's Zeitschrift*, volume 11 (1911), page 465). This method of illustration promises to be particularly expedient; it represents not only figures in a plane, but also stereometric ones by means of virtual images of space curves and surfaces. The apparatus has been patented, and is provided with a number of models for representation. A detailed prospectus is furnished with the machine.

14. The principle of correspondence introduced by Study (*American Journal*, volume 29, pages 116 ff.) by means of which the speers of elliptic space are associated with the point-spheres of two spheres of unit radius in euclidean space is applied by Professor Salkowski to the theory of space curves. If x_{l_1} x_{l_2}, x_{l_3}, x_{r_1}, x_{r_2}, x_{r_3} are the speer coordinates of the directed tangent of a space curve, y_{l_i}, y_{r_i}; z_{l_i}, z_{r_i} $(i = 1, 2, 3)$ those of its principal normal and binormal respectively, κ, τ the curvature and torsion, then the equations exist

$$\frac{dx_{l_i}}{ds} = \kappa y_{l_i}, \quad \frac{dx_{r_i}}{ds} = \kappa y_{r_i},$$

$$\frac{dy_{l_i}}{ds} = -\kappa x_{l_i} - (\tau - 1)\, x_{l_i}, \quad \frac{dy_{r_i}}{ds} = -\kappa x_{r_i} - (\tau + 1) z_{r_i},$$

$$\frac{dz_{l_i}}{ds} = (\tau - 1)\, y_{r_i}, \quad \frac{dz_{r_i}}{ds} = (\tau + 1) y_{r_i}.$$

These equations determine a curve of elliptic space, and, except as to a simple transformation, associate two curves c_l, c_r of euclidean space. A number of consequences of this theorem were pointed out.

16. The report on automorphic functions was presented by Professor Koebe. The most emphasis was put on the problem of uniformization, and a statement of recent progress made was given. After discussing his own publications (*Jahresbericht*, for 1906, 1907, 1910; *Göttinger Nachrichten*, 1907–1911; *Mathematische Annalen*, volumes 67, 69; *Crelle's Journal*, volumes 138, 139), the speaker pointed out the significance of recent memoirs of Poincaré, Hilbert, and others.

17. If four singular (real) points and their corresponding exponents of a linear differential equation of the second order are known, there are ∞^1 real values of the parameter, so that the circular quadrilateral has an orthogonal circle. In case of complex parameters, an infinite number of new theorems can be stated, one associated with each characteristic oscillation number. To generalize the above conditions, Professor Hilb uses the relations established by Klein for the triangle and by Ihlenberg for the quadrilateral. It is shown that the method developed by the latter becomes too complicated to be further extended. The author showed how to pass from polygons of n sides to those of $n+1$ sides by a new method. The corresponding extension in the complex field is much more complicated, but can be made. By means of the Fricke normal polygons, it is possible to overcome the difficulty imposed by the Stieltjes boundary, making the transformation of y' (or higher derivative) for y, then taking the auxiliary points into account.

18. Dr. Bieberbach reported on the contents of the third installment of Fricke-Klein, Vorlesungen über automorphe Funktionen, and made some remarks concerning the proof by means of continuity of the theorem of the closed section.

VIRGIL SNYDER.

A NEW PROOF OF THE EXISTENCE THEOREM FOR IMPLICIT FUNCTIONS.

BY PROFESSOR GILBERT AMES BLISS.

(Read before the American Mathematical Society October 28, 1911.)

THE theorem with which this paper has to do is the one which states the existence of a set of functions

$$y_i = y_i(x_1, x_2, \cdots, x_m) \qquad (i = 1, 2, \cdots, n)$$

which satisfy a system of equations of the form

$$(1) \qquad f_i(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n) = 0 \quad (i = 1, 2, \cdots, n).$$

For the case in which the functions f are only assumed to be continuous and to have continuous first derivatives, the proof seems to have been originally given by Dini.* His method is to show the existence of a solution of a single equation, and then to extend his result by mathematical induction to a system of the form given above, a plan which has been followed, with only slight alterations and improvements in form, by most writers on the theory of functions of a real variable. In a more recent paper† Goursat has applied a method of successive approximation which enabled him to do away with the assumption of the existence of the derivatives of the functions f with respect to the independent variables x.

One can hardly be dissatisfied with either of these methods of attack. It is true that when the theorem is stated as precisely as in the following paragraphs, the determination of the neighborhoods at the stage when the induction must be made is rather inelegant, but the difficulties encountered are not serious. The introduction of successive approximations is an interesting step, though it does not simplify the situation and indeed does not add generality with regard to the assumptions on the functions f. The method of Dini can in fact, by only a slight modification, be made to apply to cases where the functions do not have derivatives with respect to the variables x.

* Lezioni di Analisi infinitesimale, vol. 1, chap. 13. For historical remarks, see Osgood, Encyclopädie der mathematischen Wissenschaften, II, B 1, § 44 and footnote 30.

† *Bulletin de la Société mathématique*, vol. 31 (1903), page 185.

The proof which is given in the following paragraphs seems to have advantages in the matter of simplicity over either of the others. It applies equally well, without induction, to one or a system of equations, and requires only the initial assumptions which Goursat mentions in his paper.

Where it is possible without sacrificing clearness, the row letters f, x, y, p, a, b will be used to denote the systems

$$f = (f_1, f_2, \cdots, f_n), \qquad x = (x_1, x_2, \cdots, x_m),$$

$$y = (y_1, y_2, \cdots, y_n), \qquad a = (a_1, a_2, \cdots, a_m),$$

$$b = (b_1, b_2, \cdots, b_n), \qquad p = (a_1, a_2, \cdots, a_m; b_1, b_2, \cdots, b_n).$$

In this notation the equations (1) have the form

$$f(x; y) = 0,$$

the interpretation being that every element of f is a function of $x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n$, and every f_i is to be set equal to zero. The notations p_ϵ, a_ϵ, b_ϵ represent respectively the neighborhoods

$$|x - a| < \epsilon, |y - b| < \epsilon; \quad |x - a| < \epsilon; \quad |y - b| < \epsilon$$

of the points p, a, b.

With these notations in mind the fundamental theorem which is to be proved may be stated as follows:

Hypotheses:

1) *the functions $f(x; y)$ are continuous, and have first partial derivatives with respect to the variables y which are also continuous, in a neighborhood of the point p;*

2) $f(a; b) = 0$;

3) *the functional determinant $D = \partial(f_1, f_2, \cdots, f_n)/\partial(y_1, y_2, \cdots, y_n)$ is different from zero at p.*

Conclusions:

1) *a neighborhood p_ϵ can be found in which there corresponds to a given value x at most one solution $(x; y)$ of the equations $f(x; y) = 0$;*

2) *for any neighborhood p_ϵ with the property just described a constant $\delta \leqq \epsilon$ can be found such that every x in a_δ has associated with it a point $(x; y)$ which satisfies the equations $f(x; y) = 0$;*

3) *the functions $y(x_1, x_2, \cdots, x_m)$ so found are continuous in the region a_δ.*

For the neighborhood p_ϵ let one be chosen in which the continuity properties of the functions f are preserved. If $(x; y)$ and $(x; y')$ are two points in p_ϵ, it follows by applying Taylor's formula to the differences $f(x; y') - f(x; y)$ that

$$f_1(x; y') - f_1(x; y) = \frac{\partial f_1}{\partial y_1}(y_1' - y_1) + \cdots + \frac{\partial f_1}{\partial y_n}(y_n' - y_n),$$

. . . . . .

$$f_n(x; y') - f_n(x; y) = \frac{\partial f_n}{\partial y_1}(y_1' - y_1) + \cdots + \frac{\partial f_n}{\partial y_n}(y_n' - y_n),$$

where the arguments of the derivatives $\partial f_i/\partial y_k$ have the form $x; y + \theta_i(y' - y)$ and $0 < \theta_i < 1$. The determinant of these derivatives is different from zero when $(x; y') = (x; y) = (a; b)$, and hence must remain different from zero if p_ϵ is restricted so that in it the functional determinant D remains different from zero. It is then impossible that $(x; y)$ and $(x; y')$ should both be solutions of the equations $f(x; y) = 0$ if y is distinct from y'.

In the corresponding region b_ϵ the function

$$\varphi(a; y) = f_1^2(a; y) + f_2^2(a; y) + \cdots + f_n^2(a; y)$$

has a minimum for $y = b$, since for that value it vanishes and for every other it is positive. In particular

$$\varphi(a; \eta) - \varphi(a; b) > m > 0$$

for the *closed* set of points η forming the boundary of b_ϵ, on account of the continuity of φ, and the inequality

$$\varphi(x; \eta) - \varphi(x; b) > m$$

remains true for all values x in a suitably chosen domain a_δ. Hence for a fixed x in a_δ the minimum of $\varphi(x; y)$ is attained at a point y *interior* to b_ϵ. At such a point, however,

$$\frac{1}{2}\frac{\partial \varphi}{\partial y_1} = f_1\frac{\partial f_1}{\partial y_1} + f_2\frac{\partial f_2}{\partial y_1} + \cdots + f_n\frac{\partial f_n}{\partial y_1} = 0,$$

. . . . . .

$$\frac{1}{2}\frac{\partial \varphi}{\partial y_n} = f_1\frac{\partial f_1}{\partial y_n} + f_2\frac{\partial f_2}{\partial y_n} + \cdots + f_n\frac{\partial f_n}{\partial y_n} = 0,$$

and this can happen only when all the elements of f are zero,

since the functional determinant D is different from zero in p_ϵ. It follows that to every point x in a_δ there corresponds in p_ϵ a solution $(x; y)$ of the equations $f(x; y) = 0$.

The functions $y(x_1, x_2, \cdots, x_m)$ defined in this way over the region a_δ are all continuous. For consider the values y and $y + \Delta y$ corresponding to two points x and $x + \Delta x$. By applying Taylor's formula it follows from the relations

$$f(x; y + \Delta y) - f(x; y) = f(x; y + \Delta y) - f(x + \Delta x; y + \Delta y),$$

which are true because $(x; y)$ and $(x + \Delta x; y + \Delta y)$ both make $f = 0$, that

$$\begin{aligned} &\frac{\partial f_1}{\partial y_1}\Delta y_1 + \frac{\partial f_1}{\partial y_2}\Delta y_2 + \cdots + \frac{\partial f_1}{\partial y_n}\Delta y_n \\ &\qquad = f_1(x; y + \Delta y) - f_1(x + \Delta x; y + \Delta y), \\ &\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\frac{\partial f_n}{\partial y_1}\Delta y_1 + \frac{\partial f_n}{\partial y_2}\Delta y_2 + \cdots + \frac{\partial f_n}{\partial y_n}\Delta y_n \\ &\qquad = f_n(x; y + \Delta y) - f_n(x + \Delta x; y + \Delta y), \end{aligned} \tag{2}$$

where the arguments of the derivatives $\partial f_i/\partial y_k$ have the form $x; y + \theta_i \Delta y$ $(0 < \theta_i < 1)$. The determinant of these derivatives is different from zero on account of the way in which p_ϵ was chosen, and the second members of the equations approach zero with Δx. Hence the same must be true of the quantities Δy, and the functions $y(x_1, x_2, \cdots, x_m)$ are seen to be continuous.

A similar application of Taylor's formula leads to the conclusion that

If the functions f have derivatives of the first order with respect to x_k which are continuous in the neighborhood of p, so have also the functions $y(x_1, x_2, \cdots, x_m)$ in the region a_δ; and if the f's have all derivatives of the nth order continuous, so have the functions $y(x_1, x_2, \cdots, x_m)$.

For suppose

$$\Delta x_1 \neq 0, \quad \Delta x_2 = \Delta x_3 = \cdots = \Delta x_m = 0.$$

Then by applying Taylor's formula to the second members of

equations (2) it follows that

$$\frac{\partial f_1}{\partial y_1}\frac{\Delta y_1}{\Delta x_1}+\frac{\partial f_1}{\partial y_2}\frac{\Delta y_2}{\Delta x_1}+\cdots+\frac{\partial f_1}{\partial y_n}\frac{\Delta y_n}{\Delta x_1}+\frac{\partial f_1}{\partial x_1}=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial f_n}{\partial y_1}\frac{\Delta y_1}{\Delta x_1}+\frac{\partial f_n}{\partial y_2}\frac{\Delta y_2}{\Delta x_1}+\cdots+\frac{\partial f_n}{\partial y_n}\frac{\Delta y_n}{\Delta x_1}+\frac{\partial f_n}{\partial x_1}=0,$$

where the arguments of the derivatives $\partial f_i/\partial x_1$ have the form $x+\theta_i'\Delta x; y+\Delta y$. Hence as Δx_1 approaches zero the quotients $\Delta y_i/\Delta x_1$ approach limits $\partial y_i/\partial x_1$ which satisfy the equations

$$\frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial f_1}{\partial y_2}\frac{\partial y_2}{\partial x_1}+\cdots+\frac{\partial f_1}{\partial y_n}\frac{\partial y_n}{\partial x_1}+\frac{\partial f_1}{\partial x_1}=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (3)$$

$$\frac{\partial f_n}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial f_n}{\partial y_2}\frac{\partial y_2}{\partial x_1}+\cdots+\frac{\partial f_n}{\partial y_n}\frac{\partial y_n}{\partial x_1}+\frac{\partial f_n}{\partial x_1}=0,$$

where the arguments of the derivatives of f are now $(x; y)$. A similar consideration shows the existence of the first derivatives with respect to the variables $x_2, x_3, \cdots, x_m$. The existence of the higher derivatives follows from the observation that the solutions of equations (3) are differentiable $n-1$ times with respect to the variables x on account of the assumption that the functions f are differentiable n times.

ON A SET OF KERNELS WHOSE DETERMINANTS FORM A STURMIAN SEQUENCE.

BY MR. H. BATEMAN, M.A.

WEYL * has recently given a theorem which states that if a kernel

$$k_n(s,t)=\sum_{p,q=1}^{n}k_{pq}\Phi_p(s)\Phi_q(t) \qquad (k_{pq}=k_{qp})$$

is formed from n functions $\Phi_p(s)$ whose squares are integrable in the interval (0, 1), then the smallest positive root of the

* *Göttinger Nachrichten*, 1911, Heft 2, p. 110.

kernel

$$h_n(s, t) = k(s, t) - k_n(s, t)$$

is not greater than the $(n+1)$th positive root of $k(s, t)$.

It has occurred to me that this theorem is a particular case of the following general theorem:

Let $k(s, t)$ be a symmetric function such that

$$\int_0^1 \int_0^1 [k(s, t)]^2 ds dt$$

is convergent, $a_{pq} = a_{qp}$ $(p, q = 1, 2, \cdots, n)$ a set of constants such that the symmetrical determinant

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

is not zero, $f_1(s)$, $f_2(s)$, $\cdots$, $f_n(s)$ a set of integrable functions such that

$$\int_0^1 [f_p(s)]^2 ds$$

is convergent. Then if

$$h_n(s, t) = \frac{1}{\Delta_n} \begin{vmatrix} k(s, t) & f_1(s) & f_2(s) & \cdots & f_n(s) \\ f_1(t) & a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n(t) & a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} = \frac{F_n}{\Delta_n}$$

and if $h_{n-1}(s, t)$ is derived from $h_n(s, t)$ by omitting the last row and column in each of the determinants F_n and Δ_n, the roots of the symmetric kernel $h_{n-1}(s, t)$ will be separated by those of $h_n(s, t)$.

Let $D(\lambda)$, $D_{n-1}(\lambda)$, $D_n(\lambda)$ be the determinants of $k(s, t)$, $h_{n-1}(s, t)$, $h_n(s, t)$ respectively; then by a known formula *

$$D_n(\lambda) = \frac{D(\lambda)}{\Delta_n} \nabla_n(\lambda),$$

* *Messenger of Mathematics*, 1908, p. 179.

where

$$\nabla_n(\lambda) = \begin{vmatrix} a_{11} + \lambda\tau_{11} & \cdots & a_{1n} + \lambda\tau_{1n} \\ \cdot & \cdots & \cdot \\ a_{n1} + \lambda\tau_{n1} & \cdots & a_{nn} + \lambda\tau_{nn} \end{vmatrix},$$

$$\tau_{pq} = \tau_{qp} = \int_0^1 f_p(s)\phi_q(s)ds = \int_0^1 \phi_p(s)f_q(s)ds,$$

and $\phi_q(s)$ is the solution of the equation

$$f_q(s) = \phi_q(s) - \lambda\int_0^1 k(s,t)\phi_q(t)dt.$$

Now let A_{pq} denote the cofactor of the constituent $a_{pq} + \lambda\tau_{pq}$ in the determinant $\nabla_n(\lambda)$; then by a property of determinants

$$A_{nn}A_{n-1,n-1} - A^2_{n-1,n} = \nabla_{n-2}(\lambda)\nabla_n(\lambda),$$

where $\nabla_{n-2}(\lambda)$ is derived from $\nabla_n(\lambda)$ by omitting the last two rows and columns.

Now $A_{nn} = \nabla_{n-1}(\lambda)$, hence when $\nabla_{n-1}(\lambda)$ vanishes $\nabla_{n-2}(\lambda)$ and $\nabla_n(\lambda)$ have opposite signs. The functions

$$\nabla_1(\lambda), \quad \nabla_2(\lambda), \quad \cdots, \quad \nabla_n(\lambda)$$

therefore form a Sturmian series and it will be seen presently that the roots of $\nabla_n(\lambda)$ separate those of $\nabla_{n-1}(\lambda)$, the roots of $\nabla_{n-1}(\lambda)$ separate those of $\nabla_{n-2}(\lambda)$, and so on.

Now the roots of $\nabla_n(\lambda) = 0$ are the same as those of $D_n(\lambda) = 0$ and I have shown in a former paper* that the roots of $D_1(\lambda)$ separate those of $D(\lambda)$, hence the functions

$$D(\lambda), \quad D_1(\lambda), \quad D_2(\lambda), \quad \cdots, \quad D_n(\lambda)$$

form a sequence such that the roots of any function in the sequence separate the roots of the preceding function.

If $\overset{+}{\lambda}_1, \overset{+}{\lambda}_2, \cdots, \overset{+}{\lambda}_{n+1}$ are the positive roots of $D(\lambda)$ arranged in order of magnitude, there will be n roots of $D_1(\lambda)$ arranged singly between the gaps, $n-1$ roots of $D_2(\lambda)$ arranged between the gaps in this second set, and so on. It is clear then that there is at least one root of $D_n(\lambda)$ between $\overset{+}{\lambda}_1$ and $\overset{+}{\lambda}_{n+1}$; this includes Weyl's theorem. We have supposed that the roots of $D(\lambda)$ are all distinct, but the necessary modification for the case of multiple roots is easily introduced.

* *Cambr. Phil. Trans.*, vol. 20 (1908), p. 374. The theorem is used again on p. 182.

If the constants a_{pq} are chosen so that the determinants Δ_n are all positive, $D_{n-2}(\lambda)$ and $D_n(\lambda)$ will have opposite signs when $D_{n-1}(\lambda)$ vanishes, and so the functions

$$D(\lambda),\quad D_1(\lambda),\quad D_2(\lambda),\quad \cdots,\quad D_n(\lambda)$$

will form a Sturmian sequence.

It has been stated that the roots of the functions $\nabla_n(\lambda)$ in the Sturmian sequence separate one another. This is not always true for a Sturmian sequence when the functions are not polynomials, but it can be shown to be true in the present case, as follows. Let $g_n(s)$, $g_n(t)$ be the cofactors of the constituents $f_n(t)$, $f_n(s)$ in the determinant F_n; then from the properties of determinants

$$F_{n-1} \cdot \Delta_n - g_n(s)g_n(t) = F_n \cdot \Delta_{n-1}.$$

Dividing out by $\Delta_{n-1}\Delta_n$, we have

$$h_n(s, t) = h_{n-1}(s, t) - \frac{g_n(s)g_n(t)}{\Delta_{n-1}\Delta_n}.$$

We can now apply the theorem mentioned before to this equation and deduce that the roots of $h_{n-1}(s, t)$ are separated by those of $h_n(s, t)$, there being one root of $h_n(s, t)$ between each consecutive pair of roots of $h_{n-1}(s, t)$.

BRYN MAWR COLLEGE,
November, 1911.

ON THE CUBES OF DETERMINANTS OF THE SECOND, THIRD, AND HIGHER ORDERS.

BY PROFESSOR ROBERT E. MORITZ.

(Read before the San Francisco Section of the American Mathematical Society, April 8, 1911.)

WHILE the square of a determinant of any order may be readily expressed as a determinant of the same order, I am not aware of the existence of a correspondingly simple method by means of which the cube of any determinant may be expressed in determinant form. For a determinant of the fourth order, Δ_4, we have indeed from a well-known property of determinants

$$\Delta_4^3 \equiv \Delta_4',$$

where Δ_4' is the determinant whose constituents are the co-

factors of the corresponding constituents of Δ_4. Another isolated case may be derived from the following general theorem due to Faà di Bruno:

The determinant which has for its first row the constituents

$$x^n, \quad x^{n-1}y, \quad x^{n-2}y^2, \quad \cdots, \quad xy^{n-1}, \quad y^n$$

and whose other rows are obtained by operating on the constituents of the first row successively by the symbolic operators

$$x'\frac{d}{dx}+y'\frac{d}{dy}, \quad \frac{1}{1.2}\left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^{(2)}, \quad \cdots,$$

$$\frac{1}{n!}\left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^{(n)}$$

is a power, namely the $\frac{1}{2}n(n+1)$th power, of the determinant $xy'-x'y$. For $n=2$, this theorem yields a determinant expression for $(xy'-x'y)^3$.

I

On multiplying each constituent of the determinant of the second order

$$\Delta_2 \equiv \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

by Δ_2, we obtain

$$\Delta_2^3 \equiv \Delta_2(a_1b_2-a_2b_1)^2 = \begin{vmatrix} a_1(a_1b_2-a_2b_1) & a_2(a_1b_2-a_2b_1) \\ b_1(a_1b_2-a_2b_1) & b_2(a_1b_2-a_2b_1) \end{vmatrix},$$

and this in turn is equivalent to the bordered determinant

$$\Delta_2^3 \equiv \frac{1}{a_1b_1}\begin{vmatrix} a_1b_1 & a_1b_2-a_2b_1 & a_2b_2 \\ 0 & a_1(a_1b_2-a_2b_1) & a_2(a_1b_2-a_2b_1) \\ 0 & b_1(a_1b_2-a_2b_1) & b_2(a_1b_2-a_2b_1) \end{vmatrix}.$$

Now subtract a_1 times the first row from the second, and add b_1 times the first row to the third, then

$$\Delta_2^3 \equiv \frac{1}{a_1b_1}\begin{vmatrix} a_1b_1 & a_1b_2-a_2b_1 & a_2b_2 \\ -a_1^2b_1 & -2a_1a_2b_1 & -a_2^2b_1 \\ a_1b_1^2 & 2a_1b_1b_2 & a_1b_2^2 \end{vmatrix},$$

or, after interchanging the first and second rows and removing factors,

$$(1) \qquad \Delta_2^3 \equiv \begin{vmatrix} a_1^2 & 2a_1a_2 & a_2^2 \\ a_1b_1 & a_1b_2 + a_2b_1 & a_2b_2 \\ b_1^2 & 2b_1b_2 & b_2^2 \end{vmatrix}$$

By adding the first and third columns to the second and putting

$$a_3 = a_1 + a_2, \quad b_3 = b_1 + b_2,$$

we may write (1) in the form

$$(2) \qquad \Delta_2^3 \equiv - \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_1b_1 & a_2b_2 & a_3b_3 \\ b_1^2 & b_2^2 & b_3^2 \end{vmatrix}.$$

We have then the following

Rule for the cube of any determinant of the second order: Write down two lines of elements

$$a_1, \quad a_2, \quad a_3(= a_1 + a_2),$$
$$b_1, \quad b_2, \quad b_3(= b_1 + b_2),$$

composed respectively of the constituents of the first row (or column) and their sum, the constituents of the second row (or column) and their sum, of any determinant of the second order. Its cube is the negative of the determinant of the third order the constituents of whose rows (or columns) are respectively the squares of the elements of the first line, the products of the elements of the first line by the corresponding elements of the second line, the squares of the elements of the second line.

II.

Next consider the determinant of the third order

$$\Delta_3 \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv c_1c_2c_3 \begin{vmatrix} a_1' & a_2' & a_3' \\ b_1' & b_2' & b_3' \\ 1 & 1 & 1 \end{vmatrix},$$

where

$$a_1' = a_1/c_1, \quad a_2' = a_2/c_2, \quad a_3' = a_3/c_3, \quad b_1' = b_1/c_1, \quad \text{etc.}$$

Expressed as a determinant of the second order,

$$\Delta_3 \equiv c_1c_2c_3 \begin{vmatrix} a_2' - a_3' & a_3' - a_1' \\ b_2' - b_3' & b_3' - b_1' \end{vmatrix},$$

from which by the use of (2)

$$\Delta_3^3 \equiv -(c_1c_2c_3)^3 \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1y_1 & x_2y_2 & x_3y_3 \\ y_1^2 & y_2^2 & y_3^2 \end{vmatrix},$$

where

$$x_1 = a_2' - a_3', \quad x_2 = a_3' - a_1', \quad x_3 = a_1' - a_2',$$
$$y_1 = b_2' - b_3', \quad y_2 = b_3' - b_1', \quad y_3 = b_1' - b_2'.$$

But

$$x_1 = a_2' - a_3' = \frac{a_2}{c_2} - \frac{a_3}{c_3} = \frac{a_2c_3 - a_3c_2}{c_2c_3} = \frac{-B_1}{c_2c_3},$$

and similarly

$$x_2 = \frac{-B_2}{c_3c_1}, \quad x_3 = \frac{-B_3}{c_1c_2},$$

$$y_1 = \frac{A_1}{c_2c_3}, \quad y_2 = \frac{A_2}{c_3c_1}, \quad y_3 = \frac{A_3}{c_1c_2},$$

where the A's and B's are the cofactors of the corresponding constituents of Δ_3. Substituting these values, we obtain after a slight reduction

$$\Delta_3^3 \equiv \frac{-1}{c_1c_2c_3} \begin{vmatrix} A_1^2 & A_2^2 & A_3^2 \\ A_1B_1 & A_2B_2 & A_3B_3 \\ B_1^2 & B_2^2 & B_3^2 \end{vmatrix} \tag{3}$$

A case of special interest is that in which the c's are unity.

Thus

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 1 & 1 \end{vmatrix}^3$$

$$\equiv -\begin{vmatrix} (a_2-a_3)^2 & (a_3-a_1)^2 & (a_1-a_2)^2 \\ (a_2-a_3)(b_2-b_3) & (a_3-a_1)(b_3-b_1) & (a_1-a_2)(b_1-b_2) \\ (b_2-b_3)^2 & (b_3-b_1)^2 & (b_1-b_2)^2 \end{vmatrix}.$$

III.

Before considering the cubes of determinants of orders higher than the third, we will establish the following

THEOREM I. *Let* $\Delta_n \equiv (a_1 b_2 \cdots n_n)$ *represent any determinant of the nth order, and let S denote the substitution which replaces*

$$\begin{aligned} a_i \text{ by } (a_i k_{n+1}) &= a_i k_{n+1} - a_{n+1} k_i, \\ b_i \text{ by } (b_i k_{n+1}) &= b_i k_{n+1} - b_{n+1} k_i, \\ &\cdots\cdots\cdots\cdots \\ n_i \text{ by } (n_i k_{n+1}) &= n_i k_{n+1} - n_{n+1} k_i \end{aligned} \qquad (i = 1\ 2,\ \cdots,\ n);$$

then

$$S\Delta_n = k_{n+1}^{n-1}\Delta_{n+1} = k_{n+1}^{n-1}(a_1 b_2 \cdots n_n k_{n+1}),$$

where Δ_{n+1} *is the determinant of the* $(n+1)$*th order, formed by bordering* Δ_n *on the right by*

$$a_{n+1},\quad b_{n+1},\quad \cdots,\quad n_{n+1},$$

and on the bottom by

$$k_1,\quad k_2,\quad \cdots\quad k_n,\quad k_{n+1}.$$

Proof.—Making the substitution indicated, we have to show that

$$S\Delta_n \equiv \begin{vmatrix} (a_1k_{n+1}-a_{n+1}k_1) & (a_2k_{n+1}-a_{n+1}k_2) & \cdots & (a_nk_{n+1}-a_{n+1}k_n) \\ (b_1k_{n+1}-b_{n+1}k_1) & (b_2k_{n+1}-b_{n+1}k_2) & \cdots & (b_nk_{n+1}-b_{n+1}k_n) \\ \cdots & \cdots & \cdots & \cdots \\ (n_1k_{n+1}-n_{n+1}k_1) & (n_2k_{n+1}-n_{n+1}k_2) & \cdots & (n_nk_{n+1}-n_{n+1}k_n) \end{vmatrix}$$

$$\equiv k_{n+1}^{n-1} \begin{vmatrix} a_1 & a_2 & \cdots & a_n & a_{n+1} \\ b_1 & b_2 & \cdots & b_n & b_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n_1 & n_2 & \cdots & n_n & n_{n+1} \\ k_1 & k_2 & \cdots & k_n & k_{n+1} \end{vmatrix}.$$

But this is merely the well-known identity which results from the reduction of any determinant of the $(n+1)$th order to one of order n.

Definition.—Let us call the substitution S which transforms any determinant $\Delta_n \equiv (a_1 b_2 \cdots n_n)$ into $\Delta_{n+1} = (a_1 b_2 \cdots n_n k_{n+1})$ the *substitution belonging to* Δ_n.

THEOREM II. *If any cofactor $H_i^{(n)}$ belonging to Δ_n is transformed by the substitution belonging to Δ_n, the result of the operation is the corresponding cofactor $H_i^{(n+1)}$ belonging to Δ_{n+1} multiplied by k_{n+1}^{n-2}.*

This theorem is an immediate consequence of Theorem I, since the cofactor $H_i^{(n)}$ of Δ_n, when bordered on the right by a_{n+1}, b_{n+1}, etc., and at the bottom by k_1, k_2, etc., becomes the cofactor $H_i^{(n+1)}$ of Δ_{n+1}. Similar conclusions may be arrived at with regard to the transformation of the minors of any order.

Theorems I and II, though they appear here as immediate consequences of well-known results, enable us to derive from certain given forms and relations other forms and relations. For any identity between a determinant and its minors will, when operated on by the substitution belonging to the given determinant, give rise to another identity between a determinant of the next higher order and its minors. These theorems, therefore, permit us to proceed from relations holding for determinants of the second or third orders to corresponding relations between determinants of any order.

IV.

We now apply the theorems of the preceding section to each member of equation (3). Δ_3 being of the third order, the substitution belonging to Δ_3 is

$$S = \left\{ \begin{matrix} a_i, & b_i, & c_i \\ (a_i d_4), & (b_i d_4), & (c_i d_4) \end{matrix} \right\} \quad (i = 1, 2, 3),$$

consequently

$$S\Delta_3 = d_4^2\Delta_4,$$

where

$$\Delta_4 = (a_1b_2c_3d_4),$$

$SA_i = d_4A_i^{(4)}$, $SB_i = d_4B_i^{(4)}$, $SC_i = d_4C_i^{(4)}$ $(i = 1, 2, 3)$ where $A_i^{(4)}, B_i^{(4)}, C_i^{(4)}$, are the cofactors of the corresponding small letters in Δ_4. Finally $Sc_i = (c_id_4)$. Hence

$$S\Delta_3^3 = d_4^6\Delta_4^3 = \frac{-d_4^6}{(c_1d_4)(c_2d_4)(c_3d_4)} \begin{vmatrix} A_1^{(4)2} & A_2^{(4)2} & A_3^{(4)2} \\ A_1^{(4)}B_1^{(4)} & A_2^{(4)}B_2^{(4)} & A_3^{(4)}B_3^{(4)} \\ B_1^{(4)2} & B_2^{(4)2} & B_3^{(4)2} \end{vmatrix}$$

which, on removing the common factor d_4^6 and dropping the indices, becomes

$$(4) \qquad \Delta_4^3 \equiv \frac{-1}{(c_1d_4)(c_2d_4)(c_3d_4)} \begin{vmatrix} A_1^2 & A_2^2 & A_3^2 \\ A_1B_1 & A_2B_2 & A_3B_3 \\ B_1^2 & B_2^2 & B_3^2 \end{vmatrix}.$$

V.

The general case now offers no further difficulty. The substitution belonging to Δ_4 is

$$S = \begin{Bmatrix} a_i, & b_i, & c_i, & d_i \\ (a_ie_5), & (b_ie_5), & (c_ie_5), & (d_ie_5) \end{Bmatrix} \qquad (i = 1, 2, 3).$$

Operating on both members of (4) with S we obtain

$$(5) \qquad \Delta_5^3 \equiv \frac{-1}{(c_1d_4d_5)(c_2d_4e_5)(c_3d_4e_5)} \begin{vmatrix} A_1^2 & A_2^2 & A_3^2 \\ A_1B_1 & A_2B_2 & A_3B_3 \\ B_1^2 & B_2^2 & B_3^2 \end{vmatrix},$$

and hence by induction

$$(6) \quad \Delta_n^3 \equiv \frac{-1}{(c_1d_4\cdots n_n)(c_2d_4\cdots n_n)(c_3d_4\cdots n_n)} \begin{vmatrix} A_1^2 & A_2^2 & A_3^2 \\ A_1B_1 & A_2B_2 & A_3B_3 \\ B_1^2 & B_2^2 & B_3^2 \end{vmatrix},$$

where the large letters are the cofactors of the corresponding small letters in Δ_n. It will be noticed that each member of (6) is of degree $3n$, as it should be.

UNIVERSITY OF WASHINGTON,
March, 1911.

NOTE ON THE MAXIMAL CYCLIC SUBGROUPS OF A GROUP OF ORDER p^m.

BY PROFESSOR G. A. MILLER.

(Read before the San Francisco Section of the American Mathematical Society, October 28, 1911.)

IF H is any non-invariant subgroup of a group G of order p^m, p being any prime number, it is well known that H is transformed into itself by at least one of its conjugates under G and hence by operators which are not contained in H.* If H is cyclic and not contained in a larger cyclic subgroup of G, it is said to be a *maximal cyclic subgroup of* G. In what follows we shall establish the

THEOREM: *A necessary and sufficient condition that every maximal cyclic subgroup of order p^a in a group G of order p^m, $m > 3$, is transformed into itself by no more than p^{a+1} operators of G is that G contains one and only one cyclic subgroup of order p^{m-1}.*

If we combine with this theorem some well-known properties of the groups of order p^m which contain operators of order p^{m-1}, it results that there are only three non-cyclic groups of order p^m which have the property that each of their maximal cyclic subgroups of order p^a is transformed into itself by only p^{a+1} operators of the group. These three groups are the three non-cyclic groups of order 2^m which involve one and only one cyclic subgroup of order 2^{m-1}.

To prove the theorem in question, we shall assume that G does not involve any operator of order p^{m-1}, since the groups of order p^m which contain operators of order p^{m-1} are so well known. We shall also assume in what follows that G satisfies the condition that each one of its maximal cyclic subgroups of order p^a is transformed into itself by exactly p^{a+1} operators of G, p^a being the order of any one of the maximal cyclic subgroup of G.

* Cf. *American Journal of Mathematics*, vol. 23 (1901), p. 173.

If H is a maximal cyclic subgroup of order p^a contained in G, it must have p^m/p^{a+1} conjugates under G and these conjugates must involve

$$\frac{p^m}{p^{a+1}}(p^a - p^{a-1}) = p^m\left(\frac{1}{p} - \frac{1}{p^2}\right)$$

distinct operators of order p^a. Since any complete set of conjugates of a non-cyclic group of order p^m is contained in a subgroup of order p^{m-1}, it results that these $p^m\left(\frac{1}{p} - \frac{1}{p^2}\right)$ operators of order p^a in G must generate a subgroup of order p^{m-1}. This subgroup includes a subgroup K of order p^{m-2} which does not involve any generator of the cyclic subgroups which are conjugate with H, but it is composed of all the other operators of the given subgroup of order p^{m-1}. Since G involves $p^m - p^{m-1}$ operators which are generators of maximal cyclic subgroups but are not found in the given subgroup of order p^m, it results that G *contains exactly* $p+1$ *distinct conjugate sets of maximal cyclic subgroups.*

As each of these distinct sets generates a subgroup of order p^{m-1} which involves K, it results that K is a characteristic subgroup of G which gives rise to an abelian quotient group of order p^2 and of type (1, 1). As all the operators of K are powers of larger operators which are not in K, it results that K cannot be cyclic. In fact, if K were cyclic, G would involve operators of order p^{m-1}, which is contrary to the assumption made above.

As K is non-cyclic it must involve more than one subgroup of order p^{m-3}, and the operators which are common to all of its subgroups of this order must constitute a characteristic subgroup with respect to which its quotient group is abelian and of type $(1, 1, 1, \cdots)$. Hence it results that G must have an invariant subgroup of index p^4 which gives rise to a quotient group such that the operators of K correspond to one of its non-cyclic subgroups of order p^2.

The operators of K which correspond to operators of order p in this quotient group cannot be powers of operators of higher order in K and hence they must all be pth powers of operators of G which are not in K. That is, this quotient group of order p^4 must be such that each of its operators of order p corresponding to K is a power of an operator of order p^2 in the rest of

this quotient group. This implies that this quotient group is abelian and of type (2, 2) when $p = 2$, and when $p > 2$ it must contain at least p invariant cyclic subgroups of order p^2. As this is contrary to the fact that G contains $p + 1$ conjugate sets which involve generating operators of its maximal cyclic subgroups, we have proved that we arrive at an absurdity by assuming that G does not involve any operator of order p^{m-1}, $m > 3$.

When $p > 2$ there are only two non-cyclic groups of order p^m which involve operators of order p^{m-1}, $m > 3$, and each of these clearly contains maximal cyclic subgroups of order p^a which are transformed into themselves by more than p^{a+1} operators of G. Hence it results that the three non-cyclic groups of order 2^m which were considered above in the second paragraph are the only non-cyclic groups of order p^m in which every maximal cyclic subgroup is transformed into itself by at most p times as many operators of the group as there are operators in this maximal subgroup. This completes the proof of the theorem in question, and hence we can assume that *every non-cyclic group of order p^m, with the exception of the three of order 2^m which involve one and only one cyclic subgroup of order 2^{m-1}, contains at least one maximal cyclic subgroup of order p^a which is transformed into itself by more than p^{a+1} operators of the group.*

UNIVERSITY OF ILLINOIS.

AN EXPRESSION FOR THE GENERAL TERM OF A RECURRING SERIES.

BY PROFESSOR TSURUICHI HAYASHI.

PROFESSOR Arthur Ranum has given in the BULLETIN, volume 17, No. 9, June, 1911, pages 457–461, an explicit form of the general term of a recurring series rationally in terms of the first few terms and the constants of the scale of relation. I will give here another more explicit and more convenient form without demonstration.

Let $u_0 + u_1 + u_2 + \ldots + u_n + \ldots$ be any recurring series of order n, and let

$$u_m = a_1 u_{m-1} + a_2 u_{m-2} + \cdots + a_n u_{m-n} \qquad (m \geqq n)$$

be its scale of relation. Then for any value of m not less than n,

$$u_m = \sum_{\nu=0}^{\nu=n-1} u_\nu A_\nu,$$

where

$$A_\nu = \sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \cdot \frac{(\alpha_1+\alpha_2+\cdots+\alpha_n-1)!}{\alpha_1!\,\alpha_2!\cdots\alpha_n!} \cdot (\alpha_n+\alpha_{n-1}+\cdots+\alpha_{n-\nu}),$$

the summation extending over all the positive integral and zero values of $\alpha_1, \alpha_2, \cdots, \alpha_n$, for which

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + n\alpha_n = m - \nu.$$

Since the formula is true when $m = n$, and assuming it to be true for the values $n + l$, $n + l - 1, \cdots, n+1$, n of m, we can prove it by mathematical induction to be true for the next value $n + l + 1$, without much difficulty, unless $l < n$. Even for the case $l<n$, a slight attention leads us to the same result.

More generally, it may also be shown that for any value of m greater than λ

$$u_m = \sum_{\nu=\lambda-n+1}^{\nu=\lambda} u_\nu A_\nu,$$

where

$$A_\nu = \sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \cdot \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_n - 1)!}{\alpha_1!\,\alpha_2!\cdots\alpha_n!} \cdot (\alpha_n + \alpha_{n-1} + \cdots + \alpha_{\lambda-\nu+1}),$$

the summation extending over all the positive integral and zero values of $\alpha_1, \alpha_2, \cdots, \alpha_n$ for which

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + n\alpha_n = m - \nu.$$

TÔHOKU IMPERIAL UNIVERSITY,
SENDAI, JAPAN.
July, 19 1.

SHORTER NOTICES.

Émile Picard. Biographie, Bibliographie analytique des Écrits. By ERNEST LEBON. Paris, Gauthier-Villars, 1910. viii+80 pp.

Paul Appell. Biographie, Bibliographie analytique des Écrits. By ERNEST LEBON. Paris, Gauthier-Villars, 1910. viii+71 pp.

THESE two new volumes of the Savants du Jour are entirely similar in plan and arrangement to their two predecessors, which were devoted to Henri Poincaré and Gaston Darboux respectively. They contain, besides the brief biographical sketches of Picard and Appell by M. Lebon, a complete classified list of their publications. The titles and references to the place of publication are in many cases followed by a brief synopsis of the content of the corresponding paper. The following notes taken from these volumes may be of some interest to readers of the BULLETIN.

Charles Émile Picard was born in Paris on July 24, 1856. He entered the École Normale in 1874 and received his doctorate in 1877. During his three years at the École Normale he was inspired by Darboux; under the guidance of the latter he produced his first memoir (presented to the Academy of Sciences, January, 1877) and his doctor's dissertation. Nearly all of his subsequent publications, however, relate closely to pure analysis; even his famous researches on algebraic surfaces are properly classified as pure analysis. In 1879, at the early age of 23 years, he published the two important theorems on integral functions that bear his name. These investigations attracted the attention of Hermite and led to a long and intimate friendship which resulted in 1881 in the marriage of Picard and the daughter of Hermite. After a short experience in teaching at Toulouse he was recalled to Paris in 1881, where he has remained ever since. He was elected to the Academy of Sciences in 1889, and was its president in 1910. He was awarded the Prix Poncelet in 1886 and the Grand Prix des Sciences mathématiques in 1888 for his work on the theory of algebraic functions of two variables. His published work numbers 341 titles.

Paul Émile Appell was born in Strasbourg, September 27,

1855. He entered the École Normale in 1873 and obtained his doctorate in 1876. His first published work relates to projective geometry and shows the influence of his teacher, Chasles. His dissertation falls into this period. But beginning with 1877 he, like Picard, devoted himself almost exclusively to the field of analysis. After a brief experience in the provinces, he was recalled to Paris in 1881, where he has remained. He was elected to the Academy of Sciences in 1892. The list of his publications contains 306 titles.

J. W. YOUNG.

Vermeintliche Beweise des Fermatschen Satzes. Besprochen von A. FLECK, PH. MAENNSCHEN, O. PERRON. *Archiv der Mathematik und Physik*, Bände 14–18.

Über das letzte Fermatsche Theorem. Von BENNO LIND. *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Heft $XXVI_2$, pp. 21–65, 1910.

THE age of circle squarers had hardly come to an end (one still meets sporadic cases here and there) when a new period, the age of the "Fermatists," arose. This genus suddenly received a tremendous boom through the "Wolfskehlsche Preisstiftung" by which a prize of 100,000 Marks is offered to him who first proves the great theorem of Fermat. And in its wake there have come a host who do not have the least ambition to add anything to mathematical knowledge but merely lust after the prize money. Their number will doubtlessly reach many thousands within the 99 years for which the prize is established.

The editors of the *Archiv* have opened their pages to a discussion of the proofs that might be submitted; and it is the pleasant (?) duty of Messrs. Fleck, Maennschen, and Perron to peruse these and point out to each writer the errors which lurk in his work. Of the 111 attempts at a proof that are discussed to date in the *Archiv*, none is successful and not one adds anything which may be of value in bringing the solution of the problem nearer completion. Fermat stated in a marginal note that *the equation*

$$x^n + y^n = z^n \tag{1}$$

has no solution in integers, for all values of n which are greater than 2, and that he possessed a most wonderful proof of this

theorem; but his proof, if such existed, has never been discovered. Each of these latter-day Fermatists believes that he has found a like remarkable proof, and is ready to grasp the prize, only to have his hopes shattered by an exposure of his errors. Some of them, nothing daunted, have made a second and even a third attempt, only to fail each time. What, besides the prize, has driven these engineers, teachers, captains of artillery, building inspectors, doctors, book-keepers, preachers, apothecaries, judges, merchants, cabinet ministers, and representatives of many other vocations and of many nationalities, to waste time, paper, and ink on the worthless stuff they have turned out? The reviewer almost believes it is because they consider the problem as a sort of challenge to their intelligence — it is stated so clearly and in such few words — it sounds so familiar — just a little extension of the Pythagorean theorem — and it asks for a solution in integers, the kind of numbers we played with in our childhood days. Carried away by these impulses, they forget that such mathematicians as Abel, Cauchy, Euler, Legendre, Kummer, and a score of others have attempted the problem before them; they do not consider that they lack the subtlety of mind, the "Scharfsinn" which the subject requires; they do not even think it worth while to find out what progress has been made along this line by previous writers, but each one starts afresh and attempts, by mighty path-breaking methods, to arrive at the goal within the compass of 50 pages or of 2 pages or even on a post-card.

And wherein do these Fermatists fail — what is the nature of their errors? Disregarding completely such puerile algebraic blunders as are usually attributed to our college freshmen, almost all the errors may be classified under one of the following types: 1) Assumption of the converse of a theorem where only the direct theorem has been proved; 2) omission of steps in the proof of a theorem whose proof in turn is either impossible or as difficult as the original theorem; 3) assumption that because certain expressions satisfy an equation, these are necessarily the only ones that do so; 4) reasoning in a circle; 5) drawing conclusions for general values from theorems which have only been proven for a particular value; 6) confusion of algebraic and numerical factoring; 7) ignorance of simple factor theorems or divisibility of numbers; 8) errors in elementary theory of congruences; 9) assumption of particular forms of the unknown

quantities; 10) assumption without proof that a certain complicated expression of fractional form which has been set up must be an integer. These types of errors are about equally numerous — the first five are errors in logic, the last five are more closely related to the theory of numbers. May the *Archiv* continue its good work in the exposure of these pseudo-proofs. Let us hope that the editors will not grow tired, within the next 96 years, of their self-imposed task, which does have its compensations on the humorous side if on no other.

Lind, in his monograph, does not attempt to prove Fermat's theorem. It is his aim to give a complete algebraic survey in simplified form of everything that has been done by elementary methods towards a solution of the problem, to point out the directions which might lead to further developments, and to give a historical review of the problem since the time of Fermat. Though not complete, still his collection of the material from so many varied sources and its presentation as a whole is very noteworthy. His work is divided into four parts. Part 1 gives the algebraic analysis of the problem by elementary methods. It contains a host of theorems, with their proof, which have been deduced concerning the relations that must exist between x, y, z, and n if the equation (1) is to hold. We first have the well-known Abel equations expressing x, y, z and the three very important functions of the form

$$\frac{x^n + y^n}{x + y}, \quad \frac{z^n - y^n}{z - y}, \quad \frac{z^n - x^n}{z - x}$$

in terms of the nth powers of 6 quantities $a, b, c, \alpha, \beta, \gamma$. These relations are the starting point of all the later investigations.

Lind includes some original work in this part and in part 4, and here he comes to grief. Fleck discusses this portion of the monograph in the *Archiv*, volume 16, 1st Heft (1910), and in reply to Lind's objections to some of Fleck's criticisms, the latter goes into more detail in volume 18, 1st Heft (1911). And so we have another exhibit in the *Archiv's* gallery of exposures. Several of the theorems which Lind gives as his own in part 1 are false in themselves or their proofs are false. In part 4, he attempts to make a most important addition to the development of the problem, viz., that the equation (1) has no solution in integers x, y, z prime to n, and that in case

$n = 6m + 1$ one of these must be divisible by $3n^2$. Unfortunately, he bases his proof upon two of his theorems dealing only with the most elementary theory of congruences (in particular, if $x \equiv a \pmod 3$, then $x^n \equiv a^n \pmod{3^2}$ for all odd primes n), which Fleck points out are both false, and thus Lind's original work is mostly worthless.

Part 2 contains a bibliography of the subject, citing 183 articles. Part 3 is the historical review. The first attempt to treat Fermat's theorem in general is due to Claude Jaquemets (1651–1729), but he did not advance very far. The very important Abel equations were first set up by Barlow (1811), and then independently by Abel, Kummer, Legendre, Lindemann, F. Lucas, and Stäckel. These men also established a large number of other formulas and theorems. One of the most important theorems that Abel stated without proof was, that if equation (1) is to have integral solutions, then none of the quantities x, y, z, $x + y$, $z - y$, etc., can be a prime. In regard to this theorem, Talbot, in 1857, proved that the two larger of the unknowns cannot be primes. It was not until 1905 that Sauer showed that the smallest of the unknowns cannot be a prime. We should add here that Cauchy, Lebesgue, Liouville, and Gruenert have also arrived at very many interesting results.

As far as the proof of the impossibility of equation (1) in integers for special values of n is concerned, Euler in 1738 gave a proof for $n = 4$, and 22 years later for $n = 3$. The case $n = 5$ is an interesting one, for in 1825 Dirichlet showed that if equation (1) has solutions, then one of the unknowns must be divisible by 5, and that this unknown cannot be odd; Legendre followed this by showing (by the same method) that this unknown cannot be even, and thus completed the case for $n = 5$. In 1840 Lamé proved the case $n = 7$. Many others have given proofs covering these cases. In 1823 Legendre showed that if (1) has solutions, then for all $n < 100$ one of the unknowns must be divisible by n; but it remained for Dickson (*Quarterly Journal*, volume 40 (1908)) to prove that (1) has no integral solutions *all prime to* n for every $n < 6{,}857$, and for the larger primes $< 7{,}000$. But the year 1847 brought the richest harvest for the problem, for then Kummer, with the help of the theory of prime ideals proved that (1) has no integral solutions for *all values of* n which are odd primes and which are contained as factors in the numerators of the first $\frac{1}{2}(n - 3)$

Bernoulli numbers — this includes all odd primes < 100, with the exception of 37, 59, 67. Ten years later he extended this to another series of exponents including these three exceptional numbers, so that the Fermat theorem was proven for all values of $n > 2$ and $\leqq 100$. Kummer received the Paris prize of 1850 for his beautiful work.

In 1909, an award of 1000 Marks was made from the Wolfskehl foundation to G. Wieferich for his proof that Fermat's equation has no solutions all prime to n unless $2^{n-1} \equiv 1 \bmod n^2$. And here the problem rests.

JOSEPH LIPKE.

Solid Geometry. By H. E. SLAUGHT and N. J. LENNES. Allyn and Bacon, Boston, 1911. vi+190 pp.

THIS book follows the plane geometry of the textbook series of the authors. It is divided into seven chapters entitled lines and planes in space, prisms and cylinders, pyramids and cones, regular and similar polyhedrons, the sphere, variable geometric magnitudes, and theory of limits.

The logical phase of the development of solid geometry, as here treated, is a great improvement over that usually found in our textbooks. Many of the more fundamental principles are formally stated as axioms. The first striking example of this is Axiom III: "If two planes have a point in common, then they have at least another point in common." This fundamental theorem of three-dimensional geometry has usually been kept as obscure as possible. In all, ten axioms are thus stated.

A brief treatment of sines, cosines, and tangents, and a few theorems on the projection of lines and of areas are introduced. Some of the theorems on trihedral angles are deferred to the chapter on the sphere, where they are related to the theory of spherical triangles. Euler's theorem is stated without proof, the usual faulty proof being inserted as an exercise in which the error in the proof is to be shown. The definition of polar spherical triangles is made completely. The proof of the theorem that the shortest path between two points on a sphere is the arc of a great circle joining the points is made to depend on the concept of the length of a curve on a sphere as the limit of the sum of the lengths of small arcs of great circles — a somewhat different notion from the limit of the sum of lengths of the chords, which has been previously used in the book for the length of a curve. In the chapter on variable geometric

magnitudes, graphs are used to show the relation between volume, surface, and edge of a regular tetrahedron, cube, etc. The final chapter on limits is excellent, and introduces many new ideas not usually presented.

The usual faulty proof of the theorem on the volume of an oblique prism is retained. The minor errors are not numerous, but the following have been noticed: On page 51, line 16, the word cylinder is used where prism is meant. On page 85, line 8 from the bottom, d is used in place of $\frac{1}{2}sd$.

The book is very teachable, and taken altogether is a marked improvement over the usual text.

F. W. OWENS.

First Course in Algebra. By H. E. HAWKES, W. A. LUBY, and F. C. TOUTON. Ginn and Company. vii+334 pp.

THE purpose of the authors as stated in the preface, namely, "to build up a text book thoroughly modern, scientifically exact, teachable and suited to the needs and to the ability of the boy and girl of fourteen," has been in a large measure accomplished. The topics and problems, with a few exceptions which will be noted later, seem to have been chosen with excellent judgment. The idea of reasoning with symbols instead of numbers is introduced gradually but insistently; transposition is explained by means of addition and subtraction and the student is taught the actual use of equations before the term equation is defined at all. The lists of examples in factoring, in linear and quadratic equations, and several other topics, are sufficiently varied and extensive to give the student a thorough drill in elementary mathematical reasoning and manipulation. The authors have made good their intention as stated in the preface to use clear and exact English throughout the book. Typographical errors are few, but we have noted the following: on page 139, line 5, read fraction instead of fractions; replace 3 by $\sqrt{3}$ in the answer to example 1, page 321. We would suggest the use of the word may instead of should in line 7, page 211. The sentence beginning in line 6, page 262, is spoiled somewhat by the presence of the two words graph and figure. It would seem better to use a capital G in the last word on page 263, and similarly in example 18, page 264. One might question the value of asking in example 21, page 322, for a proof that the product of conjugate imaginary numbers is real, after conjugates have been defined on the preceding page

as imaginaries whose product is real. Well-founded objection might be made to examples 9, 10, and 11 on page 212. They involve a principle of physics with which the first-year algebra student will quite certainly be unfamiliar. The force of this objection depends of course on the degree to which we wish to encourage the student to accept principles or facts on a merely plausible explanation, which is probably all that would ever be given in class. The authors have in a number of cases introduced principles quite beyond the province of the book to discuss and have then built problems on these principles. While this is a matter on which there may be room for argument, it is the reviewer's opinion that there is no such dearth of desirable material for problems for the first year's work in algebra that one need bring in totally strange ideas from physics and geometry in order to have plenty. The subject of graphs is emphasized to a degree that makes certain parts of the book resemble an intuitional analytic geometry. The proof that a straight line may be defined by an equation of the first degree is not too difficult for the student after a very brief study of plane geometry. Until he is prepared for some such proof would it not be wiser to let the graph alone? We feel that when the student is beginning a subject that is essentially logical, all unnecessary appeals to his intuition should be avoided.

J. V. McKELVEY.

CORRECTIONS.

On page 66 of the current volume of the BULLETIN the writer gave the following theorem:

If on the interval ab

$$\sum_{n=0}^{\infty} U_n(x) = f(x),$$

$U_i(x)$ $(i = 0, \cdots, \infty)$ and $f(x)$ being continuous on the interval ab, then in order that $\sum_{n=0}^{\infty} U_n(x)$ shall be uniformly convergent on the interval ab it is necessary and sufficient that for any x_i on ab, any arbitrary number δ (however small), and any arbitrary integer N there is an integer $N'(i, \delta, N)$ greater than N, which satisfies the following condition:

On the interval $x_i - \delta$, $x_i + \delta$ there is a value x_i' of x such that for every $N_i'' > N'(i, \delta, N)$

$$\left| \sum_{n=0}^{N'(i,\delta,N)} U_n(x_i') - f(x_i') \right| \geqq \left| \sum_{n=0}^{N_i''} U_n(x) - f(x) \right|$$

for every value of x on the interval $x_i - \delta$, $x_i + \delta$ which also lies on ab.

In its present form this theorem is incorrect so far as it relates to a sufficient condition. The error consists in making N_i depend on δ. Instead we must require that an N_i exists such that for any δ subsequently chosen the conditions given shall hold. In giving the proof of the sufficiency of the condition this is the form in which the hypothesis was used.

However the condition as modified is not necessary. Hence we have failed to obtain a necessary and sufficient condition. The theorem as stated in its original form gives a necessary condition, while the amended form gives a sufficient condition. Professor Osgood very kindly called the writer's attention to this error.

N. J. LENNES.

IN my paper on "Invariant conditions that a p-ary form may have multiple linear factors" in the BULLETIN, volume 17, No. 9 (June, 1911) the condition $R_3 = 0$ in equations (7), page 455, should be replaced by $R_3' = 0$, where R_3' is the resultant of $x_2^2\phi_{1x_1/x_2}$ and the greatest common divisor (linear) of $x_2^3\phi_{0x_1/x_2}$ and $x_2^2\phi_0{}'_{x_1/x_2}$.

O. E. GLENN.

THE following typographical corrections should be made in my review of Holton's Shop Mathematics, in the December number of the BULLETIN:

Page 138, footnote. For Warnier read Warner.

Page 140. In the formulas $P_1 = \pi/P$, $P = 16$, $N = DP \cos Y$, for P read p.

C. N. HASKINS.

NOTES.

At the annual meeting of the London mathematical society held on November 9 the following papers were read: By J. E. Campbell, "The invariants of the linear partial differential equation of the second order, having two independent variables"; by H. Hilton, "On invariants of a canonical substitution"; by C. T. Bennett, "The system of lines on a cubic surface"; by G. H. Hardy and J. E. Littlewood, "The relations between Borel's and Cesaro's methods of summation"; by W. P. Milne, "A method of establishing the 27-line configuration on a cubic surface"; by H. Bateman, "Mathematical analogues of mental phenomena."

At the meeting of the Edinburgh Mathematical Society on November 10, the following papers were read: By W. P. Milne, "The system of cubic curves circumscribing two triangles and apolar to them"; by W. P. Milne, "An easy geometrical representation of the sextic covariant of a binary quartic"; by T. Hugh Miller, "On the numerical calculation of natural logarithms."

At the meeting of December 8 the following papers were read: By William Peddie: "The mechanical solution of an n-ic equation, and exhibition of a model for the determination of the roots of a cubic; by D. M. Y. Sommerville: "Einstein's principle of relativity in its kinematical aspect" and "Note on Legendre's and Bertrand's proofs of the parallel postulate by infinite areas."

The officers of the society for the year 1911–1912 are: president, D. M. Y. Sommerville; vice-president, A. G. Burgess; secretary, Peter Cowrie.

The first congress of Russian professors of mathematics will be held at St. Petersburg January 9–16, 1912.

A new journal, entitled the "*Vector*," has been established at Warsaw. It will be devoted to mathematical and physical science in general and especially to questions of method and pedagogy.

The publishing house of Gauthier-Villars in Paris announces the following mathematical works in press: Works of Hermite,

volume 3; Works of Laplace, volume 14; Stoffaes, Course in higher mathematics, volume 2, curves and surfaces and differential equations. Five further volumes in the Borel series of monographs on the theory of functions, volume 3 of Goursat's Cour d'analyse and volume 4 of Picard's Traité are nearly ready for the press.

THE seventh edition of the catalogue of mathematical models has just appeared from the firm of Martin Schilling, Leipzig. It is arranged in two parts, the first giving descriptions of the several series of models and a scientific explanation of each, the second containing an illustrated price catalogue arranged according to subject matter. The catalogue occupies 172 pages, and contains descriptions of 377 models in stock.

The two latest additions are an apparatus for generating a hyperboloid of revolution by turning one line about another skew to it, by Professor K. DOEHLEMANN, of the University of Munich, and a plaster model of a surface of constant latitude, by Dr. E. MEISSNER of the technical school at Zürich.

THE publication of an Introduction to the Lie theory of one-parameter groups, with applications to the solution of differential equations, by Dr. A. COHEN, of John Hopkins University, is announced by D. C. Heath and Company.

THE Jean Reynaud prize of ten thousand francs, awarded by the Paris academy of sciences every five years, has been bestowed this year on Professor EMILE PICARD, for his contributions to mathematics.

AT the annual meeting of the Royal society of London held in November a royal medal was awarded to Professor GEORGE CHRYSTAL, of the University of Edinburgh, for his researches in mathematics and physics, and a Copley medal was awarded to Professor Sir G. H. DARWIN, of the University of Cambridge, for his contributions to the theory of astronomical evolution.

PROFESSOR E. W. HOBSON, of the University of Cambridge, has been elected a member of the academy of sciences of Halle.

AT the University of Liège M. J. DERUYTS has been appointed professor of analysis and M. E. FAIRON professor of geometry, algebra, and methodology.

DR. TRAYNARD, of the University of Lille, has been appointed professor of differential and integral calculus at the University of Besançon.

DR. MAILLET has been appointed professor of mathematics and mechanics at the Ecole des ponts et chaussées at Paris.

THE College entrance examination board has appointed two groups of examiners in mathematics for 1912. The first will prepare the questions in algebra, and is composed of H. E. HAWKES, chairman, C. R. MACINNES, and W. A. FRANCIS; the second, composed of V. SNYDER, chairman, P. F. SMITH, and J. T. RORER, is to prepare the questions in geometry and trigonometry.

AT the University of Washington, Mr. G. I. GAVETT has been promoted to an assistant professorship of mathematics. Miss A. D. BIDDLE and Mr. JOHN WHITMORE have been appointed instructors in mathematics.

Mr. J. S. LOCKNER, of Lehigh University, has been appointed instructor in mathematics at the Case school of applied science at Cleveland, Ohio.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARMBRECHT (A.). Zum grossen Fermatschen Satz! 2te, verbesserte Auflage. Dresden, Köhler, 1911. 8vo. 4 pp. M. 0.50

BONOLA (R.). Non-Euclidean geometry. A critical and historical study of its development. Authorized English translation, with additional appendices, by H. S. Carslaw. Chicago, Open Court Pub. Co., 1911. 10+258 pp.

BORGHINO (G. N.). Metodo generale di estrazione delle radici e di soluzione delle equazioni, con numerose tavole per l'estrazione delle radici e la soluzione delle equazioni fino al 10° grado. Introduzione alla logica matematica. Torino, Paravia, 1911. 8vo. 192 pp. L. 2.50

BOTTO (A.). Soluzione geometrica del problema relativo alla duplicazione del cubo. Torino, Paravia, 1911. 8vo. 13 pp.

BROMWICH (T. J. I'a). Elementary integrals. London, Macmillan, 1911. 8vo. Sewed. 1s.

BURALI-FORTI (C.). Sopra una formola generale per la trasformazione di integrali di omografie vettoriali. (Nota.) Torino, Bona, 1911. 8vo. 23 pp.

CALEGARI (A.). Brevi nozioni di calcolo infinitesimale. Livorno, Giusti, 1911. 16mo. 139 pp. L. 2.00

CARSLAW (H. S.). See BONOLA (R.).

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Edition française. Tome I, volume 4: Statistique, assurances, économie politique. Fascicule 4. Leipzig, Teubner, 1911. 8vo. pp. 481–640. M. 6.00

ENDER (A.) Die konformen Raumtransformationen. 1ter Teil. (Progr.) Waidhofen, 1911.

EULERI (L.). Opera omnia. Sub auspiciis societatis scientarium naturalium helveticæ edenda curaverunt F. Rudio, A. Krazer, P. Stäckel. Series III: Opera physica, miscellanea, epistolæ. Vol. III, Pars 1: Dioptrica. Ed. Emil Cherbuliez. Vol. 1: Continens librum primum, libri secundi sectionem primam et secundam. Leipzig, Teubner, 1911. 8vo. 510 pp. M. 24.00

FRANKL (M. W.). Der Verhältniskalkül. Ein Beitrag zur logischen Algorithmik und zur Gegenstandstheorie. (Progr.) 1911. 8vo. 23 pp.

FUCHS (C. A.). Die Aehnlichkeitsbeziehungen zweier und mehrerer Kreise. (Progr.) Komotau, 1911. 8vo. 23 pp.

GIDÁLY (R.). Die Hauptmethoden der Konstruktion einer Fläche zweiter Ordnung aus neun gegebenen Punkten. (Progr.) Wiener-Neustadt, 1911. 8vo. 40 pp.

GLÄSER (K.). Ebene Katakausten. (Progr.) Linz, 1911. 8vo. 7 pp.

GRÜNBERGER (E.). Das Schnittwinkelproblem dreier Kreise. (Progr.) Plan, 1911. 8vo. 9 pp.

GRUNERT (W.). Beiträge zur Theorie der Bewegungen und Umlegungen im Raum. (Diss.) Königsberg, 1911. 8vo. 46 pp.

HÄMMERLE (H.). Die Stirlingsche Formel. (Progr.) Krems, 1911. 8vo. 20 pp.

HAUPT (O.). Untersuchungen über Oszillations-theoreme. (Diss.) Würzburg, 1911. 8vo. 50 pp.

HECKEL (J.). Über trigonometrische Reihen. (Progr.) Reichenberg, 1911. 8vo. 31 pp.

HENDERSON (A.). The twenty-seven lines upon the cubic surface. London, Cambridge University Press, 1911. 8vo. 108 pp. 4s. 6d.

HERZ (N.). Philosophische Konzeption und mathematische Analyse in der Weltbetrachtung. Bologna, Zanichelli, 1911. 8vo. 32 pp.

HOLZ (R.). Uber die Bestimmung der Schnittpunkte von Geraden und Kegelschnitten. (Zyklographische Grundgedanken.) (Progr.) Teplitz-Schönau, 1911. 8vo. 8 pp.

JANKCUTZ (L.). Über die in zwei Kegelschnitte zerfallende Durchdringungskurve zweier Flächen zweiten Grades. (Progr.) Klagenfurt, 1911. 8vo. 16 pp.

LINDBORG (G.). Elementerna af derivatkalkylen jämte problemsamling. Uppsala, Appelberg, 1910. 8vo. 4+68 pp. Kr. 2.00

LUSSAN (É.). Essai de démonstration générale du théorème de Fermat. Paris, Gauthier-Villars, 1911.

MELICHAR (P. J.). Určeni geom. místa pólu řezu svazku plochy 2 st. vzhledem ku svazku paprskovému. (Progr.) Kroměříži, 1911. 8vo. 6 pp.

MICELI (L.). Condizioni di divisibilità di un numero N per un numero a, scritti nel sistema di numerazione indo-arabica. Matera, Conti, 1911. 8vo. 8 pp.

MILLAR (W. J.). Inledning till differential- och integralkalkylen med exempel på dess anvendning inom mekaniken. Översat från 8e engelska upplaga av E. Johnson. Stockholm, Norstedt, 1911. 8vo. 71 pp. Kr. 1.90

MILNE (J. J.). An elementary treatise on cross ratio geometry, with historical notes. London, Cambridge University Press, 1911. 8vo. 312 pp. 6s.

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ONDRACZEK (H.). Der Dirichlet'sche Satz über die linearen Funktionen im Zahlenkörper der Determinante "+5." (Progr.) Wien, 1911. 8vo. 45 pp.

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PEZZO (P. DEL). Principi di geometria proiettiva: lezioni dettate nell' università di Napoli nell' anno 1910–1911. Napoli, Alvano, 1911. 8vo. 360 pp.

POINCARÉ (H.). Calcul des probabilités. Rédaction de A. Quiquet. 2e édition. Paris, Gauthier-Villars, 1912. 8vo. 340 pp. Fr. 12.00

RAJAKOWITSCH (J.). Elemente der Funktionenlehre. (Fortsetzung.) Geschrieben für Schüler. (Progr.) Triest, 1911. 8vo. 13 pp.

RANUCCI (D. N.). Risoluzione dell' equazione $x^n - Ay^n = \pm 1$, con una nuova dimostrazione dell' ultimo teorema di Fermat. Roma, Pontificia, 1911. 8vo. 22 pp. L. 2.00

RULF (F.). Behandlung des Plückerschen Konoides auf Grund einer neuen Definition. (Progr.) Wien, 1911. 8vo. 18 pp.

SCHMIDT (J.). Der Infinitesimalkalkül. (Progr.) Wien, 1911. 8vo. 28 pp.

SCHWACHA (P. B.). Über die Existenz und Anzahl der Wurzeln der Kongruenz $\sum_{i=0}^{n} c_i x^i \equiv 0 \pmod{m}$. (Progr.) Wilhering, 1911. 8vo. 30 pp.

STAHL (H.). Abriss einer Theorie der algebraischen Funktionen einer Veränderlichen in neuer Fassung. Nachgelassene Schrift in Verbindung mit E. Löffler herausgegeben von M. Noether. Leipzig, Teubner, 1911. 8vo. 4+103 pp. M. 5.00

STUDY (E.). Vorlesungen über ausgewählte Gegenstände der Geometrie. Erstes Heft: Ebene analytische Kurven und zu ihnen gehörige Abbildungen. Leipzig, Teubner, 1911. 8vo. 4+126 pp. M. 4.80

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BRUMBY (A. G.). Key to elementary practical geometry. (Normal Tutorial series.) London, Normal Press, 1911. 8vo. Sewed. 1s.

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COMBEROUSSE (C. DE). Cours de mathématiques à l'usage des candidats à l'Ecole polytechnique, etc. 5e édition, revue et augmentée. Tome 1er, 2e partie; algèbre élémentaire. Paris, Gauthier-Villars, 1911. 8vo. 295 pp. Fr. 6.00

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DENIS-GUIBERT (H.). Etude sur les cinquante pas géométriques dans nos colonies. Mayenne, Colin, 1911. 8vo. 140 pp.

DIETSCH (C.). See SIEVERT (H.).

ECKHARDT (E.). Éléments de trigonométrie rectiligne. Hoogstraeten, Van Hoof-Roelaus, 1904. 8vo. 35 pp. Fr. 0.50

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GENAU (A.) und KRÖMEKE (J.). Geometrie für das Lyzeum und die gymnasialen Kurse der Mädchenbildungsanstalten. Im Anschluss an die Raumlehre für Lehrerinnenbildungsanstalten und höhere Mädchenschulen von A. Genau. Leipzig, Reisland, 1911. 8vo. 8+167 pp. Cloth. M. 2.00

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GRÉVY (A.). Géométrie plane, conforme au programme du 27 juillét 1905 à l'usage des élèves du 1er cycle *A* (classes de quatrième *A* et troisième *A*.) Paris, Vuibert, 1912. 16mo. 8+283 pp.

HAGSTRÖM (K. L.). Matematiska uppgifter i studentexamen för latingymnasiet med anvisningar och svar. Linköping, Carlson, 1911. 8vo. 16 pp. Kr. 0.25

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HEDSTRÖM (J. S.) och RENDAHL (C.). Trigonometri för läroverken. Stockholm, Bonnier, 1911. 8vo. 6+121 pp. Kr. 2.50

KOMMERELL (B.) und KOMMERELL (K.). Analytische Geometrie. Für den Schulgebrauch bearbeitet. 1ter Teil. Tübingen, Laupp, 1911. 8vo. 8+192 pp. M. 2.40

KRATZ (M.). See FOCKE.

KRÖMECKE (J.). See GENAU (A.).

LÄROBOK i räkning och geometri för armén. II: Algebra och geometri. Stockholm, Norstedt, 1910. 8vo. 62+61 pp. Kr. 1.25

LAUE (J. A. C.). See CHAMPION (H. H.).

LESSER (O.). See SCHWAB (K.).

LINK (T.). See VOLLKOMMER (M.).

MANUEL d'algèbre et de trigonométrie, par une réunion de professeurs. Paris, Poussielgue, 1911. 18mo. 8+216 pp.

MARTELLO (D.). Esercizi di aritmetica e geometria, per la quarta classe elementare. Milano, Nugoli 1911. 16mo. 160 pp. L. 0.80

MEISER (W.). Lösungen zu Aufgaben aus der algebraischen Analysis, nach J. Lieblein bearbeitet. (Fortsetzung.) Nürnberg, Korn. 1911. 8vo. 3+59 pp. M. 2.00

MÖNKEMEYER (K.) und RÜSEWALD (K.). Lehr- und Übungsbuch der Mathematik für höhere Mädchenschulen. Auf Grund der neuen Lehrpläne bearbeitet. 4tes Heft. Klasse 1. Leipzig, Quelle und Meyer, 1911. 8vo. 4+80 pp. M. 1.00

MORAWETZ (J.). Vierstellige logarithmische und trigonometrische Tafeln, nebst einigen Hilfstafeln. Zum Schulgebrauch herausgegeben. Leipzig, Freytag, 1911. 8vo. 52 pp. M. 0.80

OTTO (F.), PETRI (W.) and ZIEGLER (J.). Mathematik für Lyzeen. 2ter Teil. Leipzig, Hirt und Sohn, 1911. 8vo. 6+167 pp. Cloth. M. 2.25

PERSIANI (O.). Elementi di geometria, compilati ad uso della quarta ginnasiale. Terza edizione, con notevoli aggiunte e modificazioni. Roma, Pontificia, 1911. 16mo. 102 pp. L. 0.80

PETRI (W.). See OTTO (F.).

PORTA (F.). I complementi di matematica per l'ammissione alla reale accademia militare e alla reale accademia navale. 5a edizione. 8vo. 295 pp. L. 5.00

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SCHMEHL (C.). Lehrbuch der ebenen Trigonometrie für höhere Lehranstalten. Giessen, Roth, 1911. 8vo. 6+152 pp. M. 2.50

SCHNEIDER (J.). Geometrie leicht gemacht! Ein Lehr- und Übungsbuch der Geometrie. Bamberg, Buchner, 1911. 8vo. 113 pp. M. 1.80

SCHWAB (K.) and LESSER (O.). Mathematisches Unterrichtswerk zum Gebrauche an den höheren Lehranstalten für die weibliche Jugend. Bearbeitet von M. Linnich. Leipzig, Freytag, 1911. 8vo. 228 pp. Cloth. M. 3.00

SIEVERT (H.) und DIETSCH (C.). Lehrbuch der Elementar-Geometrie zum Gebrauche an Mittelschulen und beim Selbstunterrichte. 1ter Teil. 1te Abteilung: Geometrie der Ebene. 5te Auflage. Leipzig, Diechert, 1912. 8vo. 6+183 pp. M. 2.40

—— Lehrbuch der Elementar-Geometrie zum Gebrauche an Mittelschulen und beim Selbstunterrichte. 1ter Teil. 2te Abteilung: Geometrie des Raumes. 3te Auflage. Leipzig, Diechert, 1912. 8vo. 6+183 pp. M. 2.40

SIMON (M.). Raumlehre für Mittelschulen. 7te Auflage, neu bearbeitet von W. Wilke. Leipzig, Hirt und Sohn, 1911. 8vo. 190 pp. Halfcloth. M. 2.00

SOLUTIONS des exercices et problèmes du cours supérieur de géométrie; par une réunion de professeurs. Paris, Poussielgue, 1911. 16mo. 6+433 pp.

STERNECK (R. v.). Der mathematische Unterricht an den Universitäten: (Berichte über den mathematischen Unterricht in Österreich. Veranlasst durch die internationale mathematische Unterrichtskommission. 7tes Heft.) Wien, Hölder, 1911. 8vo. 6+50 pp. M. 1.20

SVENSSON (E. B.). Fuldständiga lösningar och svar till de algebraiska studentuppgifterna på reallinjen. 1896–1910. Stockholm, Bergvall, 1910. 8vo. 168 pp. Kr. 3.00

UNTERRICHT, Der mathematische, in Schweden. Berichte der schwedischen Abteilung der internationalen mathematischen Unterrichtskommission. Herausgegeben von H. v. Koch und E. Göransson. Stockholm, Fritze, 1911. 8vo. 7+215 pp. M. 3.40

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